NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ON FRACTALS

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

2001

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of Doctor of Philosophy

9 August, 2001

TL D900

Abstract

The study of nonlinear partial differential equations on fractals is a burgeoning inter-disciplinary topic, allowing dynamic properties on fractals to be investigated. In this thesis we will investigate nonlinear PDEs of three basic types on bounded and unbounded fractals.

We first review the definition of post-critically finite (p.c.f.) self-similar fractals with regular harmonic structure. A Dirichlet form exists on such a fractal; thus we may define a weak version of the Laplacian. The Sobolev-type inequality, established on p.c.f. self-similar fractals satisfying the separation condition, plays a crucial rôle in the analysis of PDEs on p.c.f. self-similar fractals. We use the classical approach to study the linear eigenvalue problem on p.c.f. self-similar fractals, which depends on the Sobolev-type inequality. Fundamental solutions such as Green's function, wave propagator and heat kernel are then explicitly expressed in terms of eigenvalues and eigenfunctions.

The main aim of the thesis is to study nonlinear PDEs on fractals. We begin with nonlinear elliptic equations on p.c.f. self-similar fractals. We prove the existence of non-trivial solutions to elliptic equations with zero Dirichlet boundary conditions using the mountain pass theorem and the saddle point theorem. For nonlinear wave equations on p.c.f. self-similar fractals, we show the existence of global solutions for appropriate initial and boundary data. We also examine blow up at finite time which may occur for certain initial data. Finally, we consider nonlinear diffusion equations on p.c.f. self-similar fractals and unbounded fractals. Using the upper-lower solution technique, we prove the global existence of solutions of the nonlinear diffusion equation with initial value and boundary conditions on p.c.f. self-similar fractals. For unbounded fractals, starting with a heat kernel satisfying certain assumptions, we prove that the diffusion equation with a nonlinear term of the form u^p possesses a global solution if the initial data is small and $p > 1 + d_s/2$, while solutions blow up if $p \leq 1 + d_s/2$ even for small initial data, where d_s is the spectral dimension of the fractal. We investigate smoothness and Hölder continuity of solutions when they exist.

Acknowledgements

During the preparation of this dissertation, I have received valuable advice and comments from Prof. Kenneth J. Falconer, my supervisor, to whom I express my warmest thanks. Without his generous help, I could not have finished this dissertation.

I thank Drs Birger Lammering and Kurt Falk for their help in improving my IT skills.

This work was financially supported by the ORS and a University of St Andrews Scholarship.

Declarations

I, Jiaxin Hu, hereby certify that this thesis, which is approximately 32000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in my previous application for a higher degree.

Date......................... Signature of candidate..........................

I was admitted as a research student in October 1998 and as a candidate for the degree of Doctor of Philosophy in October 1998; the higher study for which this is a record was carried out in the University of St Andrews between 1998 and 2001.

Date......................... Signature of candidate..........................

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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Contents.

About this dissertation.

Chapter 1 begins with the definition of post-critically finite (p.c.f) self-similar fractals developed by Kigami, and Dirichlet forms are then introduced. We establish a Sobolev-type inequality on p.c.f. self-similar fractals that possess regular harmonic structure and satisfy the separation condition. We give some examples of fractals for which an embedding inequality holds. We define a weak Laplacian by using the Dirichlet form on the fractal, and provided it is continuous, show it is to be the same as Kigami's standard Laplacian.

In Chapter 2 we establish basic properties of the the eigenvalues and eigenfunctions for the Laplacian on p.c.f. fractals using the Sobolev-type inequality of Chapter 1. The techniques are standard but we include the results for completeness.

Chapter 3 goes to construct the fundamental solutions such as heat kernels, Green's functions and wave propagators on fractals. The results of Chapter 2 are extensively exploited. These fundamental solutions are the corner stones for studying non-linear partial differential equations on fractals. They present a striking contrast to their counterparts on classical domains. The reason for this is that the spectral dimension is less than 2 in the fractal case.

Chapter 4 investigates non-linear elliptic equations on p.c.f self-similar fractals. Using the mountain pass theorem and the saddle point theorem, we prove the existence of (multiple) non-trivial solutions of non-linear elliptic equations with zero Dirichlet boundary conditions.

Chapter 5 studies nonlinear wave equations on bounded fractals. We show the global existence of a strong solution to nonlinear wave equations with initial-boundary conditions. We use the contraction principle to obtain the local existence of a strong solution, which can be extended globally by using a priori estimates obtained using the Sobolev-type inequality.

Chapter 6 concentrates on nonlinear diffusion equations on bounded fractals. We establish a maximum principle on this class of fractals. We then use the 'lower-upper solution' technique to obtain the global existence of a weak solution to the nonlinear diffusion equation with given initial conditions and zero boundary conditions.

Chapter 7 turns to nonlinear diffusion equations on more general fractals that may be unbounded. Several universal assumptions about a heat kernel are made that hold on many basic fractals. We show that there is a critical parameter that depends on the spectral dimension and determines whether non-negative solutions exist or whether 'blow-up' occurs.

CHAPTER 1

Sobolev-type inequalities on p.c.f. self-similar fractals

In this chapter we first review the definition of post-critically finite (p.c.f.) self-similar fractals introduced by Kigami. We define Dirichlet forms on p.c.f. self-similar fractals and establish a Sobolev-type inequality on p.c.f. self-similar fractals satisfying the separation condition. We give two examples of p.c.f. self-similar fractals on which the Sobolev-type inequality holds. Finally, we define the (weak) Laplacian on p.c.f. self-similar fractals.

1.1. P.c.f. self-similar fractals and Dirichlet forms

In this section we first recall some basic concepts in fractal geometry such as Hausdorff measures and Hausdorff dimensions. We then review a certain class of fractals, post-critically finite (p.c.f.) self-similar fractals, and introduce Dirichlet forms on p.c.f. fractals.

Let $n \geq 1$ be an integer and $\mathbb{R}^n$ the usual n-dimensional Euclidean space. Let V be a subset of $\mathbb{R}^n$ and $s > 0$. A finite or countable collection of subsets $\{U_i\}$ of $\mathbb{R}^n$ is called a *δ-cover* of V if $V \subset \cup_{i=1}^{\infty} U_i$ and the diameter $\operatorname{diam}(U_i)$ of U_i for all i is less than δ. For all $\delta > 0$, we define

$$\mathcal{H}^s_\delta(V) = \inf\left\{\sum_{i=1}^{\infty}\Big(\operatorname{diam}(U_i)\Big)^s : \{U_i\} \text{ is a } \delta\text{-cover of } V\right\}. \tag{1.1}$$

This infimum is increasing as $\delta \searrow 0$. Thus we define

$$\mathcal{H}^s(V) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(V). \tag{1.2}$$

It may be shown that $\mathcal{H}^s$ is a regular Borel measure on $\mathbb{R}^n$, and we term it the *s-dimensional Hausdorff measure* of V. It is easy to see from (1.1) and (1.2) that for all sets $V \subset \mathbb{R}^n$ there is a number $\dim_H(V)$, called the *Hausdorff dimension* of V, such that $\mathcal{H}^s(V) = \infty$ if $s < \dim_H(V)$ and $\mathcal{H}^s(V) = 0$ if $s > \dim_H(V)$, that is

$$\dim_H(V) = \inf\{s : \mathcal{H}^s(V) = 0\} = \sup\{s : \mathcal{H}^s(V) = \infty\}.$$

Let $N \geq 2$ be an integer. An *iterated function system* (IFS) is a family of contraction mappings $\{F_1, F_2, \cdots, F_N\}$ on $\mathbb{R}^n$, that is $F_i : \mathbb{R}^n \to \mathbb{R}^n$ $(1 \leq i \leq N)$

and

$$|F_i(x) - F_i(y)| \leq \alpha_i |x - y| \quad \text{for all } x, y \in \mathbb{R}^n, \tag{1.3}$$

where $0 < \alpha_i < 1$ and $|.|$ is the Euclidean metric. If (1.3) is replaced by equality then F_i is a *similitude*. For an IFS $\{F_1, F_2, \cdots, F_N\}$ on $\mathbb{R}^n$ there exists a unique, non-empty compact set $V \subset \mathbb{R}^n$ satisfying

$$V = \bigcup_{i=1}^{N} F_i(V),$$

see [**10**, p. 30]. Such a set V is called the *attractor* of the IFS $\{F_1, F_2, \cdots, F_N\}$; if the F_i are all similitudes, V is called a *self-similar fractal.*

An IFS $\{F_1, F_2, \cdots, F_N\}$ satisfies the *open set condition* if there exists a non-empty bounded open set $U \subset \mathbb{R}^n$ such that

$$\bigcup_{i=1}^{N} F_i(U) \subset U$$

with this union disjoint. For an IFS $\{F_1, F_2, \cdots, F_N\}$ of similitudes satisfying the open set condition there is a unique number $d_f > 0$ such that

$$\sum_{i=1}^{N} \alpha_i^{d_f} = 1.$$

Such a d_f is the Hausdorff dimension of the self-similar fractal V of the IFS $\{F_1, F_2, \cdots, F_N\}$, see [**9**, p.118-120]. Moreover, $0 < \mathcal{H}^{d_f}(V) < \infty$, see [**9**, p.118-120].

A certain class of self-similar fractals, termed post-critically finite (p.c.f.) self-similar fractals, was introduced in [**24**]. Let $\sum$ be a shift space based on $S = \{1, 2, \cdots, N\}$, that is

$$\sum = \{\omega : \omega = i_1 i_2 \cdots \text{ with } i_k \in S \text{ for all } k \in \mathbb{N}\},$$

where $\mathbb{N}$ is the collection of all positive integers. For the self-similar fractal V of an IFS $\{F_1, \cdots, F_N\}$ we define a mapping $\pi : \sum \to V$ by

$$\{\pi(\omega)\} = \bigcap_{m=1}^{\infty} F_{i_1 i_2 \cdots i_m}(V),$$

for all $\omega = i_1 i_2 \cdots \in \sum$, where

$$F_{i_1 i_2 \cdots i_m} = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_m}.$$

It is easy to see that π is well defined since $\cap_{m=1}^{\infty} F_{i_1 i_2 \cdots i_m}(V)$ is a singleton, due to the contraction property of F_i $(1 \leq i \leq N)$. Let

$$\begin{aligned}
\Lambda &= \bigcup_{\substack{i,j \in S \\ i \neq j}} F_i(V) \cap F_j(V), \\
\Pi &= \pi^{-1}(\Lambda), \\
P &= \bigcup_{k=1}^{\infty} \sigma^k(\Pi),
\end{aligned}$$

where σ is a shift map, that is $\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 \cdots$. We call V a *post-critically finite* (p.c.f.) self-similar fractal if the *post-critical set* P is finite. (Here the definition of p.c.f. self-similar fractal is more restrictive than that given in [**23**].) Let $V_0 = \pi(P)$, termed the *boundary* of V. Then V is the closure of V_* under the Euclidean metric, where $V_* = \bigcup_{m=0}^{\infty} V_m$ and $V_{m+1} = \bigcup_{i \in S} F_i(V_m)$ with $V_m \subset V_{m+1}, m \geq 0$, see [**23**, Lemma 1.3.10, p. 13].

We now give the definition of Dirichlet forms on fractals. Let V be a self-similar fractal of an IFS $\{F_1, \cdots, F_N\}$. Let μ be a regular Borel measure with support V and $0 < \mu(V) < \infty$. (Note: here and elsewhere when we say that the support of μ is V, we mean that $\mu(U) > 0$ for any open subset $U \subset V$.) Such a measure exists on V, see [**10**, p.36-38]. Let $L^2(V)$ be the usual space of square integrable functions on V with respect to μ, with the norm $\|\,.\,\|_2$. A *Dirichlet form* W on V is a non-negative, closed, Markovian and symmetric bilinear form on $\mathcal{D}(W) \times \mathcal{D}(W)$ where $\mathcal{D}(W)$ is a dense subspace of $L^2(V)$, that is

- **(Non-negative)**: $W(u,u) \geq 0$ for all $u \in \mathcal{D}(W)$.
- **(Bilinear)**: $W(\tau_1 u_1 + \tau_2 u_2, v) = \tau_1 W(u_1, v) + \tau_2 W(u_2, v)$ for all $\tau_1, \tau_2 \in \mathbb{R}$ and $u_1, u_2, v \in \mathcal{D}(W)$.
- **(Symmetric)**: $W(u,v) = W(v,u)$ for all $u, v \in \mathcal{D}(W)$.
- **(Closed)**: for a sequence $\{u_k\}$ in $\mathcal{D}(W)$ satisfying
  $$\|u_k - u_l\|_2^2 + W(u_k - u_l, u_k - u_l) \to 0, \ k, l \to \infty,$$
  there exists a $u \in \mathcal{D}(W)$ such that
  $$\|u_k - u\|_2^2 + W(u_k - u, u_k - u) \to 0, \ k \to \infty.$$
- **(Markovian)**: if $u \in \mathcal{D}(W)$, then $T_\varepsilon(u) \in \mathcal{D}(W)$ and $W(T_\varepsilon(u), T_\varepsilon(u)) \leq W(u,u)$ for all $\varepsilon > 0$, for all $T_\varepsilon : \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$ satisfing $T_\varepsilon(x) = x$ if $x \in [0,1]$ and
  $$0 \leq T_\varepsilon(x_2) - T_\varepsilon(x_1) \leq x_2 - x_1, \text{ for } x_1 < x_2.$$

The classical Dirichlet form on a smooth domain $\Omega \in \mathbb{R}^n$ is

$$W(u,v) = \frac{1}{2}\sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\, dx, \tag{1.4}$$

for $u, v \in W^{1,2}(\Omega)$, where $W^{1,2}(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for } 1 \leq i \leq n\}$ is the usual Sobolev space and $\frac{\partial u}{\partial x_i}$ the distributional derivative of u with respect to x_i. Clearly W in (1.4) is a non-negative, closed and symmetric bilinear form on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. With certain amount of effort, one can show that W in (1.4) is also Markovian, see [**17**, p.8]. Thus W is a Dirichlet form on Ω.

For a p.c.f. self-similar fractal V, a Dirichlet form may be obtained in the following way. Let $V_0 = \{p_1, p_2, \cdots, p_{n_0}\}$ be the boundary of V. Define a quadratic form $W_0 : \mathcal{D}(W_0) \times \mathcal{D}(W_0) \to \mathbb{R}$ by

$$W_0(u,v) = \frac{1}{2}\sum_{i,j=1}^{n_0} c_{ij}\, (u(p_i) - u(p_j))(v(p_i) - v(p_j)), \tag{1.5}$$

where $c_{ij} = c_{ji} \geq 0$ $(1 \leq i, j \leq n_0)$ and $u, v \in \mathcal{D}(W_0) \equiv \{u |\ u : V_0 \to \mathbb{R}\}$. We suppose that W_0 is *irreducible*, that is

$$W_0(u,u) = 0 \quad \text{if and only if } u \text{ is constant on } V_0. \tag{1.6}$$

We inductively define a quadratic form $W_{m+1} : \mathcal{D}(W_{m+1}) \times \mathcal{D}(W_{m+1}) \to \mathbb{R}$ on V_{m+1} by

$$W_{m+1}(u,v) = \sum_{i=1}^{N} r_i^{-1}\, W_m(u \circ F_i, v \circ F_i) \tag{1.7}$$

for $m \geq 0$ and $u, v \in \mathcal{D}(W_{m+1}) \equiv \{u |\ u : V_{m+1} \to \mathbb{R}\}$, where $r_i > 0$ for all $i \in S$.

For $u \in \mathcal{D}(W_0)$, we define

$$\mathbb{W}_1(u,u) = \min\{W_1(v,v) |\ v \in \mathcal{D}(W_1) \quad \text{and} \quad v|_{V_0} = u\}. \tag{1.8}$$

A p.c.f. self-similar fractal V is said to possess a *harmonic structure*, denoted by (J, r), if there exist an $n_0 \times n_0$ matrix $J = -(c_{ij})$ and a vector $r = (r_1, r_2, \cdots, r_N)$ such that

$$\mathbb{W}_1(u,u) = W_0(u,u) \quad \text{for all} \quad u \in \mathcal{D}(W_0). \tag{1.9}$$

The harmonic structure (J, r) is said to be *regular* if $r_i < 1$ for all $i \in S$, see [**24, 28**]. It is an open question whether or not a general p.c.f. self-similar fractal possesses a regular harmonic structure although a positive answer was obtained for nested fractals in [**35, 37**] (a nested fractal is a fractal of an IFS with the same contraction ratio for all F_i and additional properties [**35**]).

From now on we assume that our fractal is a p.c.f. self-similar fractal in $\mathbb{R}^n$ which possesses a regular harmonic structure, except where otherwise stated.

For $u : V_* \to \mathbb{R}$, let

$$W(u,u) = \lim_{m\to\infty} W_m(u,u), \tag{1.10}$$

(possibly $W(u,u) = \infty$). Note that (1.10) makes sense since $\{W_m(u,u)\}$ is non-decreasing in $m \geq 0$ for all $u : V_* \to \mathbb{R}$; this is because V possesses a harmonic structure and thus, for all $u : V_* \to \mathbb{R}$ and $m \geq 1$,

$$\begin{aligned} W_{m+1}(u,u) - W_m(u,u) &= \sum_{i=1}^{N} r_i^{-1} \left(W_m(u\circ F_i, u\circ F_i) - W_{m-1}(u\circ F_i, u\circ F_i) \right) \\ &\geq 0 \end{aligned}$$

if $W_m(u,u) \geq W_{m-1}(u,u)$, giving that $\{W_m\}$ is non-decreasing in m by using the fact that $W_1(u,u) \geq W_0(u,u)$ and induction. The W in (1.10) is only defined on V_*. By a continuous extension such a W may be viewed as the Dirichlet form on V with the domain $\mathcal{D}(W)$ in $C(V)$, the space of all continuous functions on V. Note that this construction of a Dirichlet form does not depend on the measure μ on V.

For a p.c.f. self-similar fractal V having a regular harmonic structure, the *effective resistence metric* $R : V_* \times V_* \to \mathbb{R}$ is given by

$$R(x,y) = \max\left\{ \frac{|u(x)-u(y)|^2}{W(u,u)} \,\middle|\, u(x) \neq u(y), W(u,u) < \infty \right\}, \tag{1.11}$$

see [**23, 27**]. Then V can be viewed as the closure of V_* under the metric R, see [**26**].

From (1.11), we see that

$$|u(x)-u(y)|^2 \leq R(x,y) W(u,u) \tag{1.12}$$

for all $x, y \in V_*$ and all $u : V_* \to \mathbb{R}$ (if $W(u,u) = \infty$, then (1.12) is obvious).

1.2. Sobolev-type inequalities.

Let V be a p.c.f. self-similar fractal of an IFS $\{F_1, \cdots, F_N\}$ of similitudes, with a regular harmonic structure (J, r). Let W be defined as in (1.10). In this section we show that the *Sobolev-type inequality*

$$|u(x)-u(y)| \leq C_1 |x-y|^{\alpha}\, W(u,u)^{1/2} \tag{1.13}$$

holds for all $x, y \in V$ and all $u \in C(V)$, provided that V further satisfies the separation condition to be stated below. Here $C_1 > 0$ and $\alpha = \ln(r_0)/(2\ln d) > 0$ with $r_0 = \max_{i\in S}\{r_i\} < 1$ and for some $d \in (0,1)$. Note that (1.13) was obtained for *nested fractals*, see [**23, 30, 31, 35**].

Recall $S = \{1, 2, \cdots, N\}$. Let $S^m = \{\omega |\ \omega = i_1 i_2 \cdots i_m, i_k \in S \text{ for } 1 \leq k \leq m\}$ and $F_\omega = F_{i_1} \circ F_{i_2} \cdots \circ F_{i_m}$ for $\omega = i_1 i_2 \cdots i_m \in S^m$.

Proposition 1.1. *Let V be a p.c.f. self-similar fractal possessing a regular harmonic structure (J,r). Then there exists a constant C_2 depending only on (J,r) such that*

$$R(x,y) \le C_2 {r_0}^m \tag{1.14}$$

for all $m \ge 0$ and all $x,y \in F_\omega(V_*)$ with $\omega \in S^m$, where $r_0 = \max_{i\in S}\{r_i\} < 1$.

Proof. See Lemma 3.3 in [**26**, p.297]. ∎

Proposition 1.2. *Let V be a p.c.f. self-similar fractal possessing a regular harmonic structure (J,r). Then there exists a constant C_3 depending only on (J,r) such that*

$$|u(x)-u(y)| \le C_3 {r_0}^{m/2}\, W(u,u)^{1/2} \tag{1.15}$$

for all $m \ge 0$ and all $x,y \in F_\omega(V)$ with $\omega \in S^m$ and all $u \in C(V)$.

Proof. Let $m \ge 0$. For $x,y \in F_\omega(V_*)$ with $\omega \in S^m$, we see that (1.15) follows immediately from (1.12) and (1.14). For $x,y \in F_\omega(V)$ with $\omega \in S^m$, there are sequences $\{x_k\}_{k\ge1}$ and $\{y_k\}_{k\ge1}$ in $F_\omega(V_*)$ such that $|x_k - x| \to 0, |y_k - y| \to 0$ as $k \to \infty$, proving (1.15) by noting that

$$|u(x_k)-u(y_k)| \le C_3 {r_0}^{m/2} W(u,u)^{1/2}. \tag{1.16}$$

and u is continuous on V. ∎

For $\omega = i_1 i_2 \cdots i_m \in S^m$, a set of the form $F_\omega(V)$ is called an *m-complex*. A p.c.f. self-similar fractal V is said to be *separated* if there exist some $\eta > 0$ and some $d \in (0,1)$ such that for all $m \ge 1$ and all $\omega_1, \omega_2 \in S^m$, we have

$$\operatorname{dist}(F_{\omega_1}(V), F_{\omega_2}(V)) \equiv \inf_{\substack{x\in F_{\omega_1}(V)\\ y\in F_{\omega_2}(V)}} \{|x-y|\} \ge \eta\, d^m \tag{1.17}$$

whenever $F_{\omega_1}(V) \cap F_{\omega_2}(V) = \emptyset$. The *separation condition* (1.17) says that the distance between any two disjoint m-complexes ($m \ge 1$) is bounded below from $\eta\, d^m$ for some $\eta > 0$.

Theorem 1.3 (Sobolev-type inequality). *Assume that V is a p.c.f. self-similar fractal that possesses a regular harmonic structure and satisfies the separation condition (1.17). Then for some C_1*

$$|u(x)-u(y)| \le C_1 |x-y|^\alpha\, W(u,u)^{1/2}$$

holds for all $x,y \in V$ and all $u \in C(V)$, where $\alpha = \ln(r_0)/(2\ln d) > 0$.

Proof. The inequality (1.13) is obvious for $u \in C(V)$ with $W(u,u) = \infty$. Suppose that $u \in C(V)$ and $W(u,u) < \infty$. Without loss of generality, we suppose that

$$|x-y| < \eta\, d. \tag{1.18}$$

Let

$$x = \pi(i_1 i_2 i_3 \cdots) \ne y = \pi(j_1 j_2 j_3 \cdots).$$

Let $1 \leq m < \infty$ be the greatest integer such that

$$F_{i_1 i_2 \cdots i_m}(V) \cap F_{j_1 j_2 \cdots j_m}(V) \neq \emptyset.$$

Such an m exists; otherwise $F_{i_1}(V) \cap F_{j_1}(V) = \emptyset$ and so $|x-y| \geq \eta\, d$ by (1.17), contradicting (1.18). Let $z \in F_{i_1 i_2 \cdots i_m}(V) \cap F_{j_1 j_2 \cdots j_m}(V)$. Noting that

$$F_{i_1 i_2 \cdots i_m i_{m+1}}(V) \cap F_{j_1 j_2 \cdots j_m j_{m+1}}(V) = \emptyset,$$

it follows from (1.17) that

$$|x-y| \geq \operatorname{dist}(F_{i_1 i_2 \cdots i_m i_{m+1}}(V), F_{j_1 j_2 \cdots j_m j_{m+1}}(V)) \geq \eta\, d^{m+1}.$$

We see from (1.15) that

$$|u(x)-u(z)| \leq C_3 {r_0}^{m/2}\, W(u,u)^{1/2} \text{ (since } x, z \in F_{i_1 i_2 \cdots i_m}(V)),$$
$$|u(z)-u(y)| \leq C_3 {r_0}^{m/2}\, W(u,u)^{1/2} \text{ (since } z, y \in F_{j_1 j_2 \cdots j_m}(V)),$$

and so,

$$\begin{aligned} |u(x)-u(y)| &\leq 2C_3 {r_0}^{m/2}\, W(u,u)^{1/2} \\ &\leq C_1\, |x-y|^{\alpha}\, W(u,u)^{1/2} \end{aligned}$$

for all $x, y \in V$ and all $u \in C(V)$ with $W(u,u) < \infty$. ■

The Sobolev-type inequality plays a crucial part in investigating nonlinear partial differential equations on bounded fractals, analogous to the classical Sobolev embedding theorems on smooth domains.

1.3. Examples.

Example 1.4. Sierpiński gasket.

Let $N \geq 2$ be an integer. Let p_i $(1 \leq i \leq N+1)$ be $N+1$ distinct points in $\mathbb{R}^N$ with $|p_i - p_j| = 1$ $(1 \leq i \neq j \leq N+1)$, where $|\,.\,|$ is the usual Euclidean metric. Let $F_i, 1 \leq i \leq N+1$ be contracting similtudes given by $F_i(x) = \frac{1}{2}(x - p_i) + p_i$ for $x \in \mathbb{R}^N$. Then there is a unique non-empty compact set V in $\mathbb{R}^N$ such that

$$V = \bigcup_{i=1}^{N+1} F_i(V).$$

The set V is the *Sierpiński gasket* in $\mathbb{R}^N$. Clearly the Sierpiński gasket is a p.c.f. self-similar fractal since V may be viewed as the closure of $V_* \equiv \bigcup_{m=0}^{\infty} V_m$, where $V_{m+1} = \bigcup_{i=1}^{N+1} F_i(V_m), m \geq 0$, with $V_0 = \{p_1, p_2, \cdots, p_{N+1}\}$. See Figure 1, for $N = 2$. Moreover, the Sierpiński gasket possesses a regular harmonic structure. To see this,

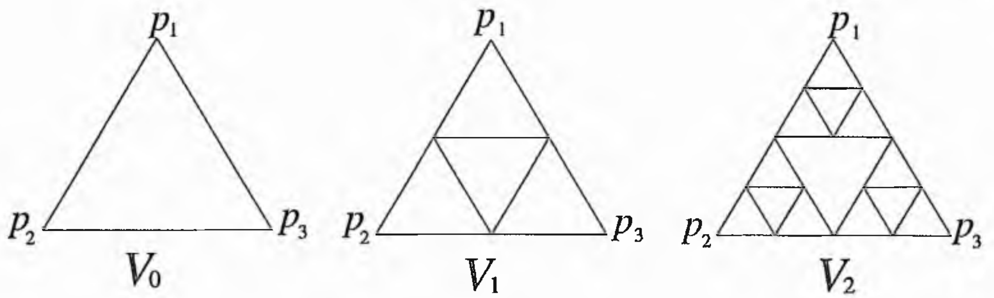

Figure 1.

we can take $c_{ij} = 1$ $(1 \le i, j \le N{+}1)$ in (1.5) and $r_i = (N{+}1)/(N{+}3)$ $(1 \le i \le N{+}1)$ in (1.7), that is

$$W_m(u,v) = \left(\frac{N+3}{N+1}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)) \tag{1.19}$$

for all $u, v : V_m \to \mathbb{R}$. Then

$$W_0(u,u) = \inf\{W_1(v,v) : v|_{V_0} = u\},$$

for all $u : V_0 \to \mathbb{R}$ and $v : V_1 \to \mathbb{R}$. The Sobolev-type inequality (1.13) holds with $\alpha = \log((N+3)/(N+1))/2\log 2$, for $r_0 = (N+1)/(N+3)$ and $d = 1/2$, see **[30]** for $N = 2$ and **[12]** for $N \ge 2$.

Example 1.5. Vicsek snowflake.

Let $F_i : \mathbb{R}^2 \to \mathbb{R}, 1 \le i \le 5$ be given by

$F_1(x) = \frac{x}{3}, \quad F_2(x) = \frac{x}{3} + \frac{2}{3}(1,0), \quad F_3(x) = \frac{x}{3} + \frac{2}{3}(1,1),$

$F_4(x) = \frac{x}{3} + \frac{2}{3}(0,1), \quad F_5(x) = \frac{x}{3} + \frac{1}{3}(1,1), \quad x \in \mathbb{R}^2.$

The *Vicsek snowflake* V in $\mathbb{R}^2$ satisfies

$$V = \bigcup_{i=1}^{5} F_i(V),$$

see **[2, 38]**, see Figure 2. The Vicsek snowflake V is a p.c.f. self-similar fractal with the boundary $V_0 = \{p_1, p_2, p_3, p_4\}$, where

$$p_1 = (0,0), \quad p_2 = (0,1), \quad p_3 = (1,1), \quad p_4 = (1,0).$$

Then we may take $c_{ij} = 1$ $(1 \le i, j \le 4)$ in (1.5) and $r_i = 1/3$ $(1 \le i \le 5)$ in (1.7), to give the Vicsek snowflake a regular harmonic structure, with the Sobolev-type inequality (1.13) holding with $\alpha = 1/2$.

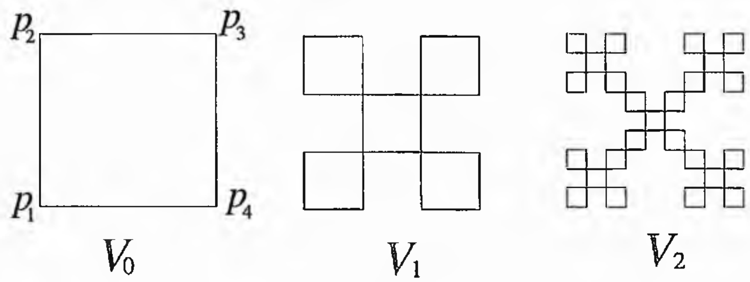

Figure 2.

The separation condition (1.17) is obvious for the fractals in Examples 1.4 and 1.5.

1.4. Laplacians on p.c.f. self-similar fractals.

Let V be a p.c.f. self-similar fractal with the boundary V_0 which possesses a regular harmonic structure. Let $W(u,u)$ be defined as in (1.10) for all $u \in C(V)$. We define

$$\mathcal{D}(W) = \{u|\ u \in C(V) \quad \text{and} \quad W(u,u) < \infty\}. \tag{1.20}$$

Let $C_0(V) = \{u \in C(V)|\ u = 0 \text{ on } V_0\}$ and $H_0^1(V) = C_0(V) \cap \mathcal{D}(W)$. Then $H_0^1(V)$ is a Hilbert space with the inner product defined as follows. Let $W(u,v)$ be given by

$$W(u,v) = \lim_{m\to\infty} W_m(u,v), \quad u,v \in \mathcal{D}(W),$$

where $W_m(u,v)$ is as in (1.7). It is easy to see using the Cauchy-Schwartz inequality that this limit exists and is finite if $u, v \in \mathcal{D}(W)$. Then the *inner product* of $u, v \in H_0^1(V)$ is $W(u,v)$, and the norm of u in $H_0^1(V)$ is $\|u\| = W(u,u)^{1/2}$. From (1.13), we immediately get

$$H_0^1(V) \quad \text{is compactly embedded into the space} \quad C_0(V) \tag{1.21}$$

by using the Arzela-Ascoli theorem. To see (1.21), let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in $H_0^1(V)$, that is there is a constant A such that $\|u_k\| \le A$ for all $k \ge 1$. From (1.13) it is easily seen that $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded in $C_0(V)$ and equicontinuous on V. Since V is compact, we use the Arzela-Ascoli theorem to get (1.21). Furthermore, taking $y \in V_0$ in (1.13) and using $u|_{V_0} = 0$, we have

Proposition 1.6.

$$\sup_{x\in V} |u(x)| \le M\ \|u\| \quad \text{for all} \quad u \in H_0^1(V), \tag{1.22}$$

where $M = C_1(\text{diam}(V))^{\alpha}$.

■

Let μ be a normalized regular Borel measure on V satisfying $\mu(V) = 1$ and $\mu(U) > 0$ for each non-empty open subset U of V. For $u \in \mathcal{D}(W)$, if there exists a function in $L^2(V)$, denoted by Δu, such that

$$W(u,v) = -\int_V \Delta u(x) v(x)\, d\mu(x), \quad \text{for all} \quad v \in H_0^1(V), \tag{1.23}$$

then we say that Δu is a *weak Laplacian* of u. (Compare the classical case when W is given by (1.4) where this follows from the divergence theorem.) Note that the definition of weak Laplacian depends on the measure μ.

Whilst we shall mainly work with the weak Laplacian, there is also a directly defined version. For $m \geq 0$, let $W_m(u,v)$ be given as in (1.7) for $u, v \in \mathcal{D}(W_m) = \{u |\ u : V_m \to \mathbb{R}\}$. Define a linear operator $L_m : \mathcal{D}(W_m) \to \mathcal{D}(W_m)$ by

$$W_m(u,v) = -\sum_{x \in V_m} (L_m u)(x)\, v(x) \quad \text{for all} \quad u, v \in \mathcal{D}(W_m). \tag{1.24}$$

It is easy to see that such an L_m is uniquely determined by the bilinear form W_m for all $m \geq 0$.

Let $p \in V_m, m \geq 0$, and $\Psi_p^m : V_* \to \mathbb{R}$ be defined by

$$\Psi_p^m(x) = \begin{cases} 1, & \text{if} \quad x = p, \\ 0, & \text{if} \quad x \neq p, x \in V_m, \end{cases} \tag{1.25}$$

$$L_k \Psi_p^m(x) = 0, \quad x \in V_* \backslash V_m \quad \text{and} \quad \text{all} \quad k \geq m. \tag{1.26}$$

Such a function Ψ_p^m is uniquely determined on V by a continuous extension from V_* to V for all $m \geq 0$, see [**24**, Theorem 3.1, p.730]. Following [**24**, Definition 6.1, p.741], we define a linear operator $\Delta_\mu^m : \mathcal{D}(W_m) \to \mathcal{D}(W_m)$ by

$$\Delta_\mu^m u(p) = \mu_{m,p}^{-1} L_m u(p), \quad p \in V_m,$$

where $\mu_{m,p} = \int_V \Psi_p^m(x)\, d\mu(x) > 0$.

Let u be a continuous function on V. If there exists a continuous function on V, denoted by $\Delta_s u$, such that

$$\lim_{m \to \infty} \sup_{p \in V \backslash V_0} \left| \Delta_\mu^m u(p) - \Delta_s u(p) \right| = 0, \tag{1.27}$$

then we say that $\Delta_s u$ is the *standard Laplacian* of u, see [**24**].

The existence of the standard Laplacian of a function $u \in H_0^1(V)$ implies the existence of the weak Laplacian Δu defined by (1.23). We state a partial converse; see Strichartz and Usher [**49, 50**] for variations on this.

Proposition 1.7. *Assume that $u \in H_0^1(V)$ and the weak Laplacian Δu is continuous in $V \backslash V_0$. Then*

$$\Delta u(x) = \Delta_s u(x), \qquad x \in V \backslash V_0. \tag{1.28}$$

Proof. We set $\Delta u(x) = \varphi(x), x \in V \backslash V_0$. Since $\varphi(x)$ is continuous in $V \backslash V_0$, the following linear problem

$$\begin{aligned} &\Delta_s z = \varphi, \qquad x \in V \backslash V_0, \\ &z\big|_{V_0} = 0, \end{aligned} \tag{1.29}$$

has a unique solution $z(x)$, which belongs to $H_0^1(V)$, see [**24**, Theorem 6.9, p.743]. Therefore,

$$W(z, v) = -\int_V \Delta_s z \, v \, d\mu, \tag{1.30}$$

for all $v \in H_0^1(V)$, see [**24**, Lemma 6.8, p.742]. From (1.23) and (1.30), we have

$$W(u - z, v) = 0, \qquad \text{for all } v \in H_0^1(V),$$

giving $u = z$ on V, and (1.28) follows immediately from (1.29). ■

There are several other approaches to defining a Laplacian on fractals, see for example [**3, 5, 6, 32**], see also Section 3.1.

CHAPTER 2

Eigenvalue problems

Let V be a p.c.f. self-similar fractal in $\mathbb{R}^n$ which possesses a regular harmonic structure and satisfies the separation condition (1.17), with the boundary V_0. Then the Sobolev-type inequality (1.13) holds for V. We shall show that there exists a sequence of eigenfunctions which forms an orthonormal basis of $L^2(V)$ and corresponds to a sequence of non-negative eigenvalues. The proof is standard, but we sketch it here for completeness.

2.1. Eigenvalue problems on p.c.f self-similar fractals

Consider the following eigenvalue problem

$$\Delta u + \lambda a(x) u = 0, \quad x \in V \backslash V_0, \tag{2.1}$$

with zero boundary conditions

$$u|_{V_0} = 0, \tag{2.2}$$

where $a : V \to [0, \infty)$. We say λ is an *eigenvalue* of (2.1) corresponding to a non-zero *eigenfunction* $\psi \in H_0^1(V)$ if

$$W(\psi, v) = \lambda \int_V a(x)\psi(x)v(x)\, d\mu(x) \tag{2.3}$$

holds for all $v \in H_0^1(V)$.

Theorem 2.1. *Let $a(x) \geq 0$ on V with $a \equiv \int_V a(x) d\mu(x) > 0$. Then (2.1) with (2.2) has a sequence of positive eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \tag{2.4}$$

where $\lambda_m \to \infty$ as $m \to \infty$. Moreover, the corresponding eigenfunctions $\{\psi_m\}_{m=1}^{\infty}$ form an orthogonal basis of the Hilbert space $H_0^1(V)$.

Proof. The technique used here is the same in spirit as on smooth domains, see for example [**36, 46**], but appealing to the Sobolev-type inequality (1.13).

We first show the existence of the eigenvalue λ_1, the smallest positive eigenvalue. By (1.22), we have that

$$\int_V a(x) u^2(x)\, d\mu(x) \leq M^2\, W(u, u) \int_V a(x)\, d\mu(x) = \beta_1 W(u, u)$$

for all $u \in H_0^1(V)$, where $\beta_1 = aM^2$. Consider the *Rayleigh quotient*

$$J(u) \equiv \frac{W(u,u)}{\int_V a(x)u^2(x)\,d\mu(x)}, \qquad u \neq 0.$$

Thus

$$J(u) \geq \frac{W(u,u)}{\beta_1 W(u,u)} = \frac{1}{\beta_1}, \qquad \text{for all } u \in H_0^1(V), u \neq 0.$$

Therefore, $J(u)$ has a positive infimum. Let

$$\mathcal{A} = \left\{ u \in H_0^1(V) : \int_V a(x)u^2(x)\,d\mu(x) = 1 \right\}.$$

We define

$$\lambda_1 = \inf_{\substack{u \in H_0^1(V) \\ u \neq 0}} J(u) = \inf_{u \in \mathcal{A}} W(u,u). \tag{2.5}$$

Then $\lambda_1 \geq \frac{1}{\beta_1} > 0$. We show that λ_1 given by (2.5) is the smallest eigenvalue of (2.1) and (2.2). To do this, we choose a minimising sequence $\{u_m\} \in \mathcal{A}$ such that

$$J(u_m) = W(u_m, u_m) \to \lambda_1. \tag{2.6}$$

The sequence $\{u_m\}$ is bounded in $H_0^1(V)$, and thus by (1.21), there exist a subsequence (also denoted by $\{u_m\}$) and a function $\psi_1 \in C_0(V)$ such that

$$\lim_{m \to \infty} \sup_{x \in V} |u_m(x) - \psi_1(x)| = 0.$$

Thus, using the Lebesgue dominated convergence theorem,

$$\int_V a(x)\psi_1^2(x)\,d\mu(x) = 1. \tag{2.7}$$

Using the fact that

$$\begin{aligned} &W(u_m - u_k, u_m - u_k) + W(u_m + u_k, u_m + u_k) \\ &\quad = 2\Big(W(u_m, u_m) + W(u_k, u_k)\Big), \end{aligned}$$

we obtain from (2.5)-(2.7) that

$$\begin{aligned} W(u_m - u_k, u_m - u_k) &\leq 2\Big(W(u_m, u_m) + W(u_k, u_k)\Big) \\ &\quad - \lambda_1 \int_V a(x)(u_m(x) + u_k(x))^2\,d\mu(x) \\ &\quad \to 4\lambda_1 - 4\lambda_1 \int_V a(x)\psi_1(x)^2\,d\mu(x) = 0, \qquad m, k \to \infty, \end{aligned}$$

which implies that $\{u_m\}$ is a Cauchy sequence in $H_0^1(V)$. Hence

$$u_m \to \psi_1 \qquad \text{in } H_0^1(V),$$

and

$$\lambda_1 = W(\psi_1, \psi_1). \tag{2.8}$$

For all $v \in H_0^1(V)$, we set

$$\begin{aligned} g(t) &= J(\psi_1 + tv) \\ &= \frac{W(\psi_1 + tv, \psi_1 + tv)}{\int_V a(x)(\psi_1(x) + tv(x))^2 \, d\mu(x)} \\ &= \frac{W(\psi_1, \psi_1) + 2tW(\psi_1, v) + t^2 W(v, v)}{\int_V a(x)\psi_1(x)^2 \, d\mu(x) + 2t \int_V a(x)\psi_1(x)v(x) \, d\mu(x) + t^2 \int_V a(x)v(x)^2 \, d\mu(x)}. \end{aligned}$$

It is easy to calculate

$$\begin{aligned} g'(0) &= \frac{2W(\psi_1, v) \int_V a(x)\psi_1(x)^2 \, d\mu(x) - 2W(\psi_1, \psi_1) \int_V a(x)\psi_1(x)v(x) \, d\mu(x)}{\{\int_V a(x)\psi_1(x)^2 \, d\mu(x)\}^2} \\ &= 2\left\{ W(\psi_1, v) - \lambda_1 \int_V a(x)\psi_1(x)v(x) \, d\mu(x) \right\} \end{aligned}$$

by virtue of (2.7) and (2.8). Since $g(t)$ reaches its minimum at $t = 0$, we have

$$W(\psi_1, v) = \lambda_1 \int_V a(x)\psi_1(x)v(x) \, d\mu(x), \qquad \text{for all } v \in H_0^1(V),$$

which says that λ_1 is an eigenvalue of (2.1) and (2.2). That λ_1 is the smallest eigenvalue follows from (2.5), since if λ is any eigenvalue, there exists $w \in H_0^1(V), w \neq 0$ such that

$$W(w, v) = \lambda \int_V a(x)w(x)v(x) \, d\mu(x) \qquad \text{for all } v \in H_0^1(V),$$

and we take $v = w$ to get

$$\lambda = \frac{W(w, w)}{\int_V a(x)w(x)^2 \, d\mu(x)} = J(w) \geq \inf_{\substack{u \in H_0^1(V) \\ u \neq 0}} J(u) = \lambda_1.$$

Next we construct eigenvalues $\lambda_2 \leq \lambda_3 \leq \cdots$. Suppose inductively that we have obtained the $m-1$ eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m-1} \ (m \geq 2),$$

with corresponding to the eigenfunctions

$$\psi_1, \psi_2, \cdots, \psi_{m-1},$$

and

$$\int_V a(x)\psi_k(x)^2 \, d\mu(x) = 1, \quad k = 1, 2, \cdots, m-1.$$

Let

$$\begin{aligned} E_{m-1} &= \operatorname{span}\{\psi_1, \cdots, \psi_{m-1}\}, \\ E_{m-1}^{\perp} &= \left\{ \psi : \ \psi \neq 0 \text{ and } \int_V a(x)\psi(x)v(x) \, d\mu(x) = 0 \text{ for all } v \in E_{m-1} \right\}. \end{aligned}$$

As above, we may prove that there exist a number λ_m and a function ψ_m satisfying

$$\psi_m \in H_0^1(V) \cap E_{m-1}^{\perp},$$

$$\int_V a(x)\psi_m(x)^2 \, d\mu(x) = 1,$$

and

$$W(\psi_m, v) = \lambda_m \int_V a(x)\psi_m(x)v(x) \, d\mu(x), \qquad \text{for all} \quad v \in H_0^1(V) \cap E_{m-1}^{\perp}.$$

Furthermore,

$$\lambda_m = J(\psi_m) = \inf_{\substack{u \in H_0^1(V) \cap E_{m-1}^{\perp} \\ u \neq 0}} J(u).$$

For all $v \in H_0^1(V)$, we may decompose v as

$$v = v_1 + v_2, \quad v_1 \in E_{m-1}, \quad v_2 \in E_{m-1}^{\perp}$$

with

$$v_1 = \sum_{k=1}^{m-1} a_k \psi_k, \qquad a_k \in \mathbb{R}.$$

Therefore,

$$\begin{aligned}
W(\psi_m, v) &= W(\psi_m, v_1) + W(\psi_m, v_2) \\
&= \sum_{k=1}^{m-1} a_k W(\psi_m, \psi_k) + W(\psi_m, v_2) \\
&= W(\psi_m, v_2) = \lambda_m \int_V a(x)\psi_m(x)v_2(x) \, d\mu(x) \\
&= \lambda_m \int_V a(x)\psi_m(x)(v_1(x) + v_2(x)) \, d\mu(x) \\
&= \lambda_m \int_V a(x)\psi_m(x)v(x) \, d\mu(x), \quad \text{for all} \quad v \in H_0^1(V).
\end{aligned}$$

Thus λ_m is an eigenvalue of (2.1) and (2.2). It is trivial to see that $\lambda_m \geq \lambda_{m-1}$.

We show that the eigenfunctions ψ_i and ψ_j corresponding to different eigenvalues $\lambda_i \neq \lambda_j$ are orthogonal in $H_0^1(V)$, that is,

$$W(\psi_i, \psi_j) = 0. \tag{2.9}$$

This follows since

$$\begin{aligned}
W(\psi_i, \psi_j) &= \lambda_i \int_V a(x)\psi_i(x)\psi_j(x) \, d\mu(x) \\
W(\psi_j, \psi_i) &= \lambda_j \int_V a(x)\psi_j(x)\psi_i(x) \, d\mu(x),
\end{aligned}$$

and so

$$(\lambda_i - \lambda_j) \int_V a(x)\psi_i(x)\psi_j(x) \, d\mu(x) = 0,$$

giving (2.9).

We claim that the dimension of the space consisting of eigenfunctions corresponding to a fixed eigenvalue is finite. To see this, suppose that there exists countably infinite sequence of eigenfunctions $\{w_k\}_{k=1}^{\infty}$ corresponding to the same eigenvalue λ, which are linearly independent in $H_0^1(V)$ and so renormalized that

$$\int_V a(x)w_k(x)^2\, d\mu(x) = 1, \qquad \text{for all} \quad k \geq 1.$$

As

$$W(w_k, w_k) = \lambda \int_V a(x)w_k(x)^2\, d\mu(x) = \lambda,$$

there exists a convergent subsequence $\{w_k\}$ in $C_0(V)$, which is impossible since

$$\begin{aligned}
&\int_V a(x)(w_l(x) - w_k(x))^2\, d\mu(x) \\
&= \int_V a(x)w_l(x)^2\, d\mu(x) + \int_V a(x)w_k(x)^2\, d\mu(x) - 2\int_V a(x)w_l(x)w_k(x)\, d\mu(x) \\
&= \int_V a(x)w_l(x)^2\, d\mu(x) + \int_V a(x)w_k(x)^2\, d\mu(x) = 2, \quad \text{if } l \neq k,
\end{aligned}$$

proving our claim.

We show next that

$$\lim_{m\to\infty} \lambda_m = +\infty. \tag{2.10}$$

To see this, suppose that there is a constant β_2 such that

$$0 < \lambda_m \leq \beta_2, \qquad \text{for all} \quad m \geq 1.$$

Assuming without loss of generality that the sequence of corresponding eigenfunctions $\{\psi_m\}$ is orthogonal, that is,

$$\int_V a(x)\psi_i(x)\psi_j(x)\, d\mu(x) = \delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} \text{for all } i, j \geq 1.$$

It follows that

$$W(\psi_m, \psi_m) = \lambda_m \int_V a(x)\psi_m(x)^2\, d\mu(x) \leq \beta_2,$$

and so there exists a convergent subsequence of $\{\psi_m\}$ in the space $C_0(V)$. But this is a contradiction since

$$\int_V a(x)(\psi_m(x) - \psi_k(x))^2\, d\mu(x) = 2,$$

giving (2.10).

Finally, we show that the eigenfunctions $\{\psi_m\}$ corresponding to $\{\lambda_m\}$ form a basis of the Hilbert space $H_0^1(V)$. Otherwise, there would exist a function $v \in H_0^1(V)$ ($v \neq 0$) which is orthogonal to all $\psi_m, m = 1, 2, \cdots$, that is,

$$W(\psi_m, v) = 0, \quad \text{for all } m \geq 1,$$

and so

$$\lambda_m \int_V a(x)\psi_m(x)v(x)\, d\mu(x) = 0 \qquad \text{for all} \quad m \geq 1.$$

It follows that

$$\int_V a(x)\psi_m(x)v(x)\, d\mu(x) = 0 \qquad \text{for all} \quad m \geq 1$$

since $\lambda_m > 0, m \geq 1$. Therefore,

$$v \in E_{m-1}^{\perp}, \qquad m \geq 2,$$

and so

$$\lambda_m \leq J(v) = \frac{W(v,v)}{\int_V a(x)v(x)^2\, d\mu(x)}, \quad m = 2, 3, \cdots . \tag{2.11}$$

But $\lambda_m \to \infty$ as $m \to \infty$, and so (2.11) is impossible for $v \neq 0$. ∎

Corollary 2.2. *There exists a non-negative eigenfunction corresponding to the first eigenvalue* λ_1.

Proof. Let ψ_1 be the eigenvalue of (2.1), (2.2) which corresponds to the first eigenvalue λ_1. We show that $|\psi_1|$ is also an eigenfunction corresponding to λ_1, thus proving Corollary 2.2. To see this, we note that

$$W(|\psi_1|, |\psi_1|) \leq W(\psi_1, \psi_1)$$

by virtue of (1.5), (1.7) and (1.10), and so, using (2.7) and (2.8),

$$J(|\psi_1|) \leq \lambda_1.$$

On the other hand, we have that

$$J(|\psi_1|) \geq \lambda_1$$

since λ_1 is the infimum of $J(u)$ over $u \in H_0^1(V), u \neq 0$. ∎

Let $a(x) \equiv 1$, we see from Theorem 2.1 that there exist a sequence of eigenfunction $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to a sequence of, also denoted by, $\{\lambda_m\}_{m=1}^{\infty}$, which will be used in later Chapters.

Since $H_0^1(V)$ is dense in $C_0(V)$ [**23, 24**], we see that $\{\varphi_m\}_{m=1}^{\infty}$ is also an orthogonal basis of $L^2(V)$. Note that

$$\|\varphi_m\|_2 = \left\{ \int_V \varphi_m(x)^2\, d\mu(x) \right\}^{1/2} = 1.$$

Thus $\{\varphi_m\}_{m=1}^{\infty}$ is an orthonormal basis of $L^2(V)$.

2.2. Further results

Let W be a Dirichlet form on V defined by (1.7) and (1.10) that is irreducible, see (1.6). Let $\mathcal{D}(W)$ be defined as in (1.20). Then $\mathcal{D}(W)$ is a Hilbert space with a norm $\|u\| = \left(W(u,u) + \int_V u(x)^2\, d\mu(x)\right)^{1/2}$. It is easy to see that $H_0^1(V) \subset \mathcal{D}(W)$. We show that (2.1) with Neumann boundary conditions also has a sequence of orthogonal eigenfunctions in $L^2(V)$, but the first eigenvalue is zero, corresponding to the constant eigenfunction. A Neumann boundary condition on p.c.f. self-similar fractals was introduced by Kigami [**24**], that is $u \in C(V)$ has a *Neumann derivative* on the boundary V_0 if $\lim\limits_{m\to\infty} L_m u(p)$ exists and is finite for all $p \in V_0$, where L_m is given as in (1.24). Such a limit is termed the Neumann derivative of u at $p \in V_0$, and we denote it by $(du)(p)$.

We consider (2.1) with zero Neumann derivative conditions

$$(du)(p) = 0, \quad p \in V_0. \tag{2.12}$$

Theorem 2.3. *The eigenvalue problem (2.1), (2.12) has a sequence of eigenfunctions $\{\psi_m\}_{m=0}^{\infty}$ in $\mathcal{D}(W)$ that corresponds to a sequence of eigenvalues $\{\lambda_m\}_{m=0}^{\infty}$ and satisfies*

$$W(\psi_m, v) = \lambda_m \int_V a(x)\psi_m(x)v(x)\, d\mu(x) \quad \text{for all } v \in \mathcal{D}(W),\ m \geq 0,$$

$$\int_V a(x)\psi_m(x)^2\, d\mu(x) = 1 \qquad \text{for all } m \geq 1.$$

Moreover, we have that $\lambda_0 = 0, \lambda_1 > 0$ with $\psi_0 = 1$ and $\psi_1 \neq 0$ on V.

Proof. The proof is similar to that given in Theorem 2.1. We omit the details. ∎

CHAPTER 3

Fundamental solutions on p.c.f. self-similar fractals

Let V be a p.c.f. self-similar fractal on which the Sobolev-type inequality (1.13) holds. In this chapter we use the eigenvalues and eigenfunctions to obtain Laplacians on V. The fundamental solutions such as the heat kernel, Green's function and wave propagator on V are also obtained. These fundamental solutions play a central part in studying nonlinear PDEs on the fractal.

3.1. Laplacians revisited and heat kernels

In this section we use eigenvalues and eigenfunctions to obtain expressions for the Laplacian and heat kernel on the p.c.f. self-similar fractal.

Let μ be a regular Borel measure on V with $0 < \mu(U) < \infty$ for any non-empty open subset U of V. Let $H_0^1(V)$ be the Hilbert space defined in Section 1.4. The Laplacian Δu of $u \in H_0^1(V)$ is defined by

$$W(u,v) = -\int_V \Delta u(x)\, v(x)\, d\mu(x) \text{ for all } v \in H_0^1(V), \tag{3.1}$$

see (1.23). From Theorem 2.1, we see that the linear problem

$$\begin{aligned} -\Delta u(x) &= \lambda u, \\ u|_{V_0} &= 0, \end{aligned} \tag{3.2}$$

has a sequence of eigenfunctions $\{\varphi_k\}$ in $H_0^1(V)$ corresponding to the positive eigenvalues $\{\lambda_k\}$ in an increasing order and forming the orthogonal basis of $H_0^1(V)$, that is $\|\varphi_k\|_2 = 1$ and

$$W(\varphi_k, v) = \lambda_k \int_V \varphi_k(x)\, v(x)\, d\mu(x) \text{ for all } v \in H_0^1(V) \text{ and } k \geq 1, \tag{3.3}$$

and

$$W(\varphi_i, \varphi_j) = \int_V \varphi_i(x)\varphi_j(x)\, d\mu(x) = 0 \quad \text{for} \quad 1 \leq i \neq j. \tag{3.4}$$

Note that $\{\varphi_k\}$ is the orthonormal basis of $L^2(V)$ since $H_0^1(V)$ is dense in $L^2(V)$.

By (3.3), we see that

$$W(\varphi_k, \varphi_k) = \lambda_k \|\varphi_k\|_2^2 = \lambda_k. \tag{3.5}$$

Since $\{\varphi_k\}_{k=1}^\infty$ is a basis of $H_0^1(V)$, if $u \in H_0^1(V)$ we have

$$u(x) = \sum_{k=1}^\infty a_k \varphi_k(x), \quad a_k \in \mathbb{R}, x \in V,$$

and so, using (3.4) and (3.5),

$$W(u,u) = \sum_{k=1}^\infty a_k^2 W(\varphi_k, \varphi_k) = \sum_{k=1}^\infty \lambda_k a_k^2.$$

Thus

$$\sum_{k=1}^\infty \lambda_k a_k^2 < \infty$$

if $u(x) = \sum_{k=1}^\infty a_k \varphi_k(x) \in H_0^1(V)$. Conversely, let $u(x) = \sum_{k=1}^\infty a_k \varphi_k(x)$ where

$$\sum_{k=1}^\infty \lambda_k a_k^2 < \infty.$$

Setting $u_m(x) = \sum_{k=1}^m a_k \varphi_k(x)$, we see that the sequence $\{u_m\}$ is Cauchy in $H_0^1(V)$, and so $u \in H_0^1(V)$. Therefore,

$$H_0^1(V) = \left\{ u(x) = \sum_{k=1}^\infty a_k \varphi_k(x) : \sum_{k=1}^\infty a_k^2 \lambda_k < \infty \right\}. \tag{3.6}$$

Let $\mathcal{D}(\Delta)$ be the *domain* of the Laplacian Δ in $H_0^1(V)$, that is

$$\mathcal{D}(\Delta) = \{u \in H_0^1(V) : \Delta u \text{ exists and } \Delta u \in L^2(V)\}.$$

For $u(x) = \sum_{k=1}^\infty a_k \varphi_k(x) \in \mathcal{D}(\Delta)$, we set $\Delta u(x) = \sum_{k=1}^\infty b_k \varphi_k(x)$. Taking $v = \varphi_i, i \geq 1$ in (3.1), we see that

$$a_i \lambda_i = W(u,v) = -\int_V \Delta u(x)\, v(x)\, d\mu(x) = -b_i,$$

and so $\Delta u = -\sum_{k=1}^\infty a_k \lambda_k \varphi_k(x)$, and $\Delta u \in L^2(V)$ if and only if $\sum_{k=1}^\infty a_k^2 \lambda_k^2 < \infty$. Therefore,

$$\mathcal{D}(\Delta) = \left\{ u(x) = \sum_{k=1}^\infty a_k \varphi_k(x) : \sum_{k=1}^\infty a_k^2 \lambda_k^2 < \infty \right\}, \tag{3.7}$$

and

$$\Delta u(x) = -\sum_{k=1}^\infty a_k \lambda_k \varphi_k(x) \quad \text{for all} \quad u(x) = \sum_{k=1}^\infty a_k \varphi_k(x) \in \mathcal{D}(\Delta). \tag{3.8}$$

Note that (3.8) gives rise to a version of the Laplacian of $u \in \mathcal{D}(\Delta)$ on the fractal V by using the eigenfunctons and eigenvalues. It is easy to see from (3.8) that for an integer $m \geq 1$, the m-th power of the Laplacian Δ is given by

$$\Delta^m u(x) = \sum_{k=1}^{\infty} (-\lambda_k)^m a_k \varphi_k(x),$$

for all $u(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \in \mathcal{D}(\Delta^m)$, the domain of Δ^m.

From (3.8), we see that $-\Delta$ is a non-negative definite and self-adjoint operator from $\mathcal{D}(\Delta)$ to $L^2(V)$. Thus there exists a unique *strongly continuous* semigroup $\{P_t, t > 0\}$ on $L^2(V)$, that is P_t satisfies

$$\lim_{t \downarrow 0} \|P_t u - u\|_2 = 0$$

for $u \in L^2(V)$. The *infinitesimal generator* of $\{P_t, t > 0\}$ is Δ, that is

$$\Delta u = \lim_{h \downarrow 0} \frac{P_h u - u}{h} \tag{3.9}$$

in the L^2-norm for $u \in \mathcal{D}(\Delta)$, see [**17**, Lemma 1.3.2, p.16]. Moreover, for $t > 0$,

$$P_t u(x) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \varphi_k(x) \quad \text{for} \quad u(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \in L^2(V). \tag{3.10}$$

The semigroup $\{P_t, t > 0\}$ has a unique kernel $K : (0, \infty) \times V \times V \to \mathbb{R}$, that is

$$P_t u(x) = \int_V K(t, x, y) u(y)\, d\mu(y) \quad \text{for all} \quad u \in L^2(V),$$

where

$$K(t, x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x) \varphi_k(y), \quad t > 0 \text{ and } x, y \in V. \tag{3.11}$$

The above K is termed the *heat kernel* in diffusion equations on V, and was identified in, for example [**2, 8**]. Another approach to obtaining the heat kernel is from the delicate probabilistic analysis, see [**4, 6, 14, 20**] (where the heat kernel is termed the *transition density*).

Let $\rho(\lambda)$ be the number of eigenvalues of (3.2) not greater than λ. A Weyl-type formula

$$C_4 \lambda^{d_s/2} \leq \rho(\lambda) \leq C_5 \lambda^{d_s/2}$$

may hold for all positive $\lambda \geq \lambda_1$, where $C_4, C_5 > 0$ and λ_1 is the first eigenvalue of (3.2), and $d_s \in [1, 2)$ is the *spectral dimension* of V. This is proved in [**18**] for the Sierpínski gasket in $\mathbb{R}^n$ and in [**28**] for a p.c.f. self-similar fractal with a regular harmonic structure. See also [**42**] for the case of a *variational fractal* though there d_s may be greater than or equal to 2. Thus there exist $\beta_3, \beta_4 > 0$ such that

$$\beta_3 k^{-2/d_s} \leq \lambda_k^{-1} \leq \beta_4 k^{-2/d_s} \tag{3.12}$$

for all $k \geq 1$.

From (1.22) and (3.5), we see that

$$\sup_{x\in V} |\varphi_k(x)| \leq M\lambda_k^{1/2}, \tag{3.13}$$

and so, using (3.12), the series on the right-hand side in (3.11) is uniformly convergent for all $x, y \in V$ and all $t \geq \delta > 0$. Thus K is well-defined on $(0,\infty) \times V \times V$. Moreover, the heat kernel K is continuous on $(0,\infty) \times V \times V$ since φ_k is continuous on V for $k \geq 1$.

We see that P_t is *Markovian* for all $t > 0$, that is

$$0 \leq P_t u \leq 1 \tag{3.14}$$

for all $u \in L^2(V)$ with $0 \leq u \leq 1$. The Markov property of the semigroup P_t is equivalent to the Markov property of W, see [17]. Here we give an alternative proof of (3.14). Let $u(x) = \sum\limits_{k=1}^{\infty} a_k\varphi_k(x) \in L^2(V)$. It follows from (3.10) that

$$\begin{aligned} 0 \leq \int_V P_t u(x)u(x)\, d\mu(x) &= \sum_{k=1}^{\infty} a_k^2 e^{-\lambda_k t} \\ &\leq \sum_{k=1}^{\infty} a_k^2 = \|u\|_2^2 \end{aligned} \tag{3.15}$$

for all $u \in L^2(V)$. If (3.14) is false, say there is a $t_0 > 0$ and $x_0 \in V$ such that $P_{t_0}u(x_0) > 1$. Since $P_{t_0}u(x)$ is continuous on V, there is a neighborhood $\mathcal{N}$ of x_0 in V such that $P_{t_0}u(x) > 1$ for $x \in \mathcal{N}$. We choose $u \in L^2(V)$ with the property that $u = 1$ on $\mathcal{N}$ and $u = 0$ on $V\backslash\mathcal{N}$, and so

$$\begin{aligned} \int_V P_{t_0}u(x)\, u(x)\, d\mu(x) - \|u\|_2^2 &= \int_V (P_{t_0}u(x) - u(x))u(x)\, d\mu(x) \\ &= \int_{\mathcal{N}} (P_{t_0}u(x) - 1)\, d\mu(x) > 0, \end{aligned}$$

which contradicts with (3.15), proving (3.14). From (3.14), we see that

$$K(t,x,y) \geq 0 \quad \text{for all} \quad t > 0 \text{ and } x, y \in V, \tag{3.16}$$

and

$$\int_V K(t,x,y)\, d\mu(y) \leq 1 \quad \text{for all} \quad t > 0 \text{ and } x \in V. \tag{3.17}$$

The only assumption on μ so far is that μ is a regular Borel measure on V with $0 < \mu(U) < \infty$ for any non-empty open subset U of V. To obtain more precise estimates on the heat kernel K, we will need μ to be a Hausdorff measure on V satisfying

$$\mu(A) = \sum_{i=1}^{N} \mu_i\, \mu(F_i^{-1}(A)) \quad \text{for all } A \subset V, \tag{3.18}$$

where $\mu_i = \mu(F_i(V))$. In general, if the IFS $\{F_i\}$ of similitudes associated with V satisfies the open set condition, we can choose $\mu_i = \alpha_i^{d_f}$ and d_f the Hausdorff dimension of V satisfying

$$\sum_{i=1}^{N} \alpha_i^{d_f} = 1,$$

so μ is normalized d_f-dimensional Hausdorff measure satisfying (3.18), see [**9, 10**]. Under the condition (3.18), we have that

$$\|u\|_2^{2+4/d_s} \leq \beta_6 \Big(W(u,u) + \|u\|_2^2 \Big) \|u\|_1^{4/d_s} \tag{3.19}$$

for all $u \in C(V)$ with $W(u,u) < \infty$, where $\beta_6 > 0$, see [**2**, Theorem 8.3, p.107]. Inequality (3.19) is termed *Nash's inequality*, which gives rise to an upper bound estimate on the heat kernel K:

$$K(t,x,x) \leq \beta_7 \left(\frac{1}{t}\right)^{d_s/2}, \quad \text{for all } x \in V \text{ and } t > 0, \tag{3.20}$$

for some $\beta_7 > 0$, see [**2**, Theorem 8.4, p.108].

3.2. Green's functions

We turn to consider Green's functions on a p.c.f. self-similar fractal V with a regular harmonic structure; thus (3.12) holds [**28**]. Green's functions have been constructed on certain classes of fractals, see for example [**24, 29**]. Here we obtain a Green's function on V in a simple way by using the eigenfunctions and eigenvalues. This version of Green's function is powerful in the theoretical study of elliptic equations on fractals.

The function $G : V \times V \to [0,\infty)$ is said to be a *Green's function* if G is symmetric on $V \times V$, that is $G(x,y) = G(y,x)$ for all $x, y \in V$, and $G(.,y) \in L^2(V)$ for all $y \in V$ and satisfies

$$-\int_V G(x,y)\Delta u(x) \, d\mu(x) = u(y) \tag{3.21}$$

for all $y \in V$ and all $u \in \mathcal{D}(\Delta)$.

Define

$$G(x,y) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \varphi_k(x)\varphi_k(y), \quad x, y \in V. \tag{3.22}$$

Theorem 3.1. *The function G defined in (3.22) is a Green's function on V.*

Proof. Clearly G is symmetric on $V \times V$. That $G(.,y) \in L^2(V)$ for all $y \in V$ follows immediately from (3.13), (3.12) and the fact that $d_s < 2$, see [**28**]. Let

$u(x) = \sum\limits_{k=1}^{\infty} a_k \varphi_k(x) \in \mathcal{D}(\Delta)$. We have from (3.22) and (3.8) that

$$-\int_V G(x,y)\Delta u(x)\, d\mu(x) = \sum_{k=1}^{\infty} a_k \varphi_k(y) = u(y),$$

giving (3.21). It remains to verify that

$$G(x,y) \geq 0 \quad \text{for} \quad x,y \in V. \tag{3.23}$$

To see this, we have from (3.11) and (3.16) that for $0 < t_1 < t_2$,

$$\begin{aligned} 0 \leq \int_{t_1}^{t_2} K(t,x,y)\, dt &= \sum_{k=1}^{\infty} \varphi_k(x)\varphi_k(y) \int_{t_1}^{t_2} e^{-\lambda_k t}\, dt \\ &= H(t_1,x,y) - H(t_2,x,y) \quad \text{for all} \quad x,y \in V, \end{aligned} \tag{3.24}$$

where

$$H(t,x,y) = \sum_{k=1}^{\infty} \frac{e^{-\lambda_k t}}{\lambda_k} \varphi_k(x)\varphi_k(y), \quad t > 0 \text{ and } x,y \in V. \tag{3.25}$$

Thus $H(t,x,y)$ is non-increasing in t for all $x,y \in V$, and so

$$H(t,x,y) \geq \lim_{s\to\infty} H(s,x,y) = 0 \quad \text{for all} \quad t > 0 \text{ and } x,y \in V, \tag{3.26}$$

showing that $G(x,y) \geq 0$ for $x,y \in V$ since $H(t,.,y) \to G(.,y)$ in the L^2-norm as $t \to 0$ for all $y \in V$. ∎

Theorem 3.1 is independent of the self-similarity property (3.18) of the measure μ. Under the condition (3.18) we can obtain a nice property on the Green's function.

Theorem 3.2. *Assume that the measure μ satisfies (3.18). Then the Green's function G is uniformly Hölder continuous on $V \times V$, that is there exists a constant $\beta_8 > 0$ such that*

$$|G(x_2,y_2) - G(x_1,y_1)| \leq \beta_8(|x_2 - x_1|^{\alpha} + |y_2 - y_1|^{\alpha}),$$

for all $x_i, y_i \in V, i = 1,2$, where α is the same as in (1.13).

Proof. We first show that there is a constant $\beta_9 > 0$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi_k^2(x)}{\lambda_k} \leq \beta_9, \quad \text{for all } x \in V. \tag{3.27}$$

To see this, let $H(t,x,y)$ be defined as in (3.25). It follows from (3.24) and (3.20) that for $t \in (0,1)$,

$$\begin{aligned} H(t,x,x) - H(1,x,x) &= \int_t^1 K(\tau,x,x)\, d\tau \\ \leq \beta_7 \int_t^1 \left(\frac{1}{\tau}\right)^{d_s/2} d\tau &= \frac{2\beta_7}{2-d_s}\left(1 - t^{1-d_s/2}\right), \end{aligned}$$

giving that, for some $\beta_9 > 0$,

$$H(t,x,x) = \sum_{k=1}^{\infty} \frac{e^{-\lambda_k t}}{\lambda_k} \varphi_k^2(x) \leq \beta_9, \quad \text{for all } t \in (0,1) \tag{3.28}$$

since, using (3.12) and (3.13),

$$H(1,x,x) = \sum_{k=1}^{\infty} \frac{e^{-\lambda_k}}{\lambda_k} \varphi_k^2(x) \leq M^2 \sum_{k=1}^{\infty} e^{-\lambda_k} < \infty.$$

Letting m be any integer, we have from (3.28) that for all $t \in (0,1)$,

$$\sum_{k=1}^{m} \frac{e^{-\lambda_k t}}{\lambda_k} \varphi_k^2(x) \leq \beta_9.$$

We let $t \to 0$ to get that

$$\sum_{k=1}^{m} \frac{\varphi_k^2(x)}{\lambda_k} \leq \beta_9$$

for all integers m, giving (3.27). Therefore, we see from (3.27) that

$$W(G(x,.),G(x,.)) = \sum_{k=1}^{\infty} \frac{\varphi_k^2(x)}{\lambda_k} \leq \beta_9 \quad \text{for all } x \in V.$$

Let

$$s_m(x,y) = \sum_{k=1}^{m} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k}, \quad m \geq 1.$$

For each $m \geq 1$ and $x \in V$, we have that $s_m(x,.)$ is continuous since $\varphi_k, k \geq 1$ is continuous on V and so $s_m(x,.) \in H_0^1(V)$. Thus, using (1.13), we see that

$$\begin{aligned} |s_m(x,y_2) - s_m(x,y_1)| &\leq C_1|y_2 - y_1|^\alpha W(s_m(x,.), s_m(x,.)) \\ &= C_1|y_2 - y_1|^\alpha \sum_{k=1}^{m} \frac{\varphi_k^2(x)}{\lambda_k} \\ &\leq \beta_9 C_1 |y_2 - y_1|^\alpha, \end{aligned}$$

for all $x, y_1, y_2 \in V$ and all $m \geq 1$. Letting $m \to \infty$, it follows that for all $x, y_1, y_2 \in V$,

$$|G(x,y_2) - G(x,y_1)| \leq \beta_8 |y_2 - y_1|^\alpha,$$

where $\beta_8 = \beta_9 C_1$. Similarly, for all $y, x_1, x_2 \in V$,

$$|G(x_2,y) - G(x_1,y)| \leq \beta_8 |x_2 - x_1|^\alpha.$$

Thus,

$$\begin{aligned} |G(x_2,y_2) - G(x_1,y_1)| &\leq |G(x_2,y_2) - G(x_1,y_2)| + |G(x_1,y_2) - G(x_1,y_1)| \\ &\leq \beta_8(|x_2 - x_1|^\alpha + |y_2 - y_1|^\alpha) \end{aligned}$$

for all $x_i, y_i \in V, i = 1,2$. ∎

Remark 3.3. Recall that the classical Laplacian Δ on a smooth domain $\Omega \in \mathbb{R}^n, n \geq 1$ admits a Green's function $G_0(x,y)$, where for $n=1$ and $\Omega = [-1,1]$,

$$G_0(x,y) = \begin{cases} \frac{1}{2}(1+x)(1-y), & x \leq y, \\ \frac{1}{2}(1+y)(1-x), & x \geq y, \end{cases}$$

and $G_0(x,y)$ behaves like $\frac{1}{2\pi}\log|x-y|$ for $n=2$ and like $\kappa|x-y|^{-n+2}, \kappa > 0$ for $n \geq 3$, if y is near to x. It is easy to see that G_0 is uniformly Lipschitz continuous on $[-1,1]\times[-1,1]$ for $n=1$ but G_0 is singular around $y=x$ for $n \geq 2$. Analogously, in our fractal case the Green's function is uniformly Hölder continuous on $V \times V$, for the fundamental reason is that the spectral dimension d_s of the fractal is less than 2.

3.3. Wave propagators

We need to construct a wave propagator by using the eigenfunctions and eigenvalues of (3.2). To do this, we first introduce Hilbert spaces on V that will also be used in studying nonlinear wave equations later on.

Definition 3.4. *For $\theta \geq 0$, define*

$$E_\theta(V) = \left\{ u(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) : \sum_{k=1}^{\infty} a_k^2 \lambda_k^{2\theta} < \infty \right\} \tag{3.29}$$

with the inner product given by

$$(u,v)_\theta = \sum_{k=1}^{\infty} a_k b_k \lambda_k^{2\theta}$$

for $u(x) = \sum_{k=1}^{\infty} a_k\varphi_k(x)$ and $v(x) = \sum_{k=1}^{\infty} b_k\varphi_k(x)$. The norm $|||\cdot|||_\theta$ in $E_\theta(V)$ is given by

$$|||u|||_\theta = \left(\sum_{k=1}^{\infty} a_k^2 \lambda_k^{2\theta} \right)^{1/2} \quad \text{for } u(x) = \sum_{k=1}^{\infty} a_k\varphi_k(x). \tag{3.30}$$

Proposition 3.5. *For all $\theta \geq 0, E_\theta(V)$ with the norm $|||\cdot|||_\theta$ is a Hilbert space. Moreover, $E_\theta(V)$ is dense in $H_0^1(V)$ for $\theta > 1/2$. In particular, $E_{1/2}(V) = H_0^1(V)$.*

Proof. Clearly $E_{1/2}(V) = H_0^1(V)$. For $\theta = 0$, we have that $E_\theta(V) = L^2(V)$. We only consider the case $\theta > 0$. Clearly $E_\theta(V)$ is a normed space for all $\theta > 0$. We show that $E_\theta(V)$ is complete for all $\theta > 0$. Suppose that $\{u_k\}_{k\geq 1}$ is a Cauchy sequence in $E_\theta(V)$, that is

$$|||u_k - u_l|||_\theta \to 0, \quad k,l \to \infty. \tag{3.31}$$

We write $u_k(x) = \sum_{m=1}^{\infty} a_m^k \varphi_m(x), a_m^k \in \mathbb{R}$. Since $\lambda_1 > 0$, we have that

$$\begin{aligned}\sup_{m\geq 1}\left|a_m^k - a_m^l\right|^2 &\leq \sup_{m\geq 1}\left(\frac{\lambda_m}{\lambda_1}\right)^{2\theta}\left|a_m^k - a_m^l\right|^2 \\ &\leq \lambda_1^{-2\theta}\sum_{m=1}^{\infty}\lambda_m^{2\theta}\left|a_m^k - a_m^l\right|^2 \\ &= \lambda_1^{-2\theta}|||u_k - u_l|||_\theta^2 \to 0, \quad k,l\to\infty.\end{aligned}$$

Therefore, for all $m \geq 1$ there exists a number a_m such that

$$\sup_{m\geq 1}\left|a_m^k - a_m\right| \to 0, \quad k\to\infty.$$

Define $u(x) = \sum_{m=1}^{\infty} a_m\varphi_m(x)$. We claim that $u \in E_\theta(V)$. To see this, note that $\{u_k\}_{k\geq 1}$ is bounded in $E_\theta(V)$ and so

$$\sum_{m=1}^{j}\left(a_m^k\right)^2\lambda_m^{2\theta} \leq \beta_{10} \quad \text{for all } j\geq 1 \tag{3.32}$$

for some β_{10} independent of j. Letting $k\to\infty$ in (3.32), it follows that

$$\sum_{m=1}^{j} a_m^2\lambda_m^{2\theta} \leq \beta_{10} \quad \text{for all } j\geq 1,$$

implying that

$$|||u|||_\theta{}^2 = \sum_{m=1}^{\infty} a_m^2\lambda_m^{2\theta} \leq \beta_{10}$$

and so $u \in E_\theta(V)$. In a similar way, we have from (3.31) that

$$|||u_k - u|||_\theta^2 \to 0, \quad k\to\infty,$$

proving that $E_\theta(V)$ is complete. It is easy to see that $E_\theta(V)$ is a Hilbert space with the inner product

$$(u,v)_\theta \equiv \sum_{m=1}^{\infty} a_m b_m \lambda_m^{2\theta} \tag{3.33}$$

for $u(x) = \sum_{m=1}^{\infty} a_m\varphi_m(x)$ and $v(x) = \sum_{m=1}^{\infty} b_m\varphi_m(x)$ in $E_\theta(V)$.

Let $u(x) = \sum_{m=1}^{\infty} a_m\varphi_m(x) \in H_0^1(V)$. From (3.6), we have that $\sum_{m=1}^{\infty} a_m^2\lambda_m < \infty$. Define $u_k(x) = \sum_{m=1}^{\infty} a_m^k\varphi_m(x)$, where

$$a_m^k = \begin{cases} a_m, & m\leq k, \\ 0, & \text{otherwise.}\end{cases}$$

For all $k \geq 1$, we have that

$$|||u_k|||_\theta^2 = \sum_{m=1}^{k} a_m^2 \lambda_m^{2\theta} < \infty,$$

and so $u_k \in E_\theta(V)$. On the other hand,

$$\|u_k - u\|^2 = \sum_{m=k+1}^{\infty} a_m^2 \lambda_m \to 0, \quad k \to \infty$$

since the series $\sum_{m=1}^{\infty} a_m^2 \lambda_m$ is convergent. Therefore, $E_\theta(V)$ is dense in $H_0^1(V)$ for $\theta > 1/2$ by noting that $E_\theta(V) \subset H_0^1(V)$ if $\theta > 1/2$. ■

Note that for all $k \geq 1$, the eigenfunctions $\varphi_k \in E_\theta(V), \theta > 0$, which means that the space $E_\theta(V)$ for all $\theta > 0$ contains an infinite dimensional subspace.

Let $\{\varphi_k\}$ be the eigenfunctions of (3.2) corresponding to eigenvalues $\{\lambda_k\}$. Let $\Phi_k(x) = \varphi_k(x)/\sqrt{\lambda_k}, x \in V, k \geq 1$. The *wave propagator* $P : [0, \infty) \times V \times V \to \mathbb{R}$ on V is given by

$$P(t, x, y) = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \Phi_k(x)\Phi_k(y), \quad t \geq 0 \quad \text{and} \quad x, y \in V, \tag{3.34}$$

which plays a central rôle in studying nonlinear wave equations on V, see Chapter 5. A slightly different version of wave propagator was addressed in [**8**]. However, it is too weak (not continuous) to be applied in studying nonlinear wave equations on fractals. We show that (3.34) is justified by the following proposition.

Proposition 3.6. *Assume that μ satisfies (3.18). Let $t_0 \geq 0$ and $y_0 \in V$ be fixed. Then $P(t_0, x, y_0) \in H_0^1(V)$, in particular, $P(t, x, y)$ is continuous on $[0, \infty) \times V \times V$. Moreover,*

$$\lim_{t \downarrow 0} \frac{\partial}{\partial t} (P(t, ., y_0), u(.)) = u(y_0) \tag{3.35}$$

for $u \in H_0^1(V)$, where $(. \,, .)$ is the inner product of $H_0^1(V)$.

Proof. Without confusion, we denote the norm of the space $E_{1/2}(V)$ by $\| \ \|$ since $E_{1/2}(V) = H_0^1(V)$. From (1.22) and (3.5), we have that

$$\sup_{x \in V} |\Phi_k(x)| \leq M \ \|\Phi_k\| = M, \tag{3.36}$$

and so, using (3.12),

$$\|P(t_0, ., y_0)\|^2 = \sum_{k=1}^{\infty} \frac{\sin^2(\sqrt{\lambda_k} t_0)}{\lambda_k} \Phi_k^2(y_0)$$

$$\leq \beta_4 M^2 \sum_{k=1}^{\infty} k^{-2/d_s} < \infty$$

for all $t_0 \geq 0, y_0 \in V$ since $d_s < 2$. Thus $P(t_0, ., y_0) \in H_0^1(V)$ for all $t_0 \geq 0$ and $y_0 \in V$; in particular, $P(t_0, x, y_0)$ is continuous on V.

Let $u = \sum_{k=1}^{\infty} a_k \varphi_k \in H_0^1(V)$, that is $\sum_{k=1}^{\infty} a_k^2 \lambda_k < \infty$. We see that

$$v(t, y_0) \equiv (P(t, ., y_0), u(.)) = \sum_{k=1}^{\infty} a_k \sin\left(\sqrt{\lambda_k} t\right) \Phi_k(y_0)$$

is uniformly convergent on $[0, \infty) \times V$ since, using (3.27),

$$\sum_{k=1}^{\infty} \left| a_k \sin\left(\sqrt{\lambda_k} t\right) \Phi_k(y_0) \right| \leq \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \Phi_k(y_0)^2\right)^{1/2} < \infty.$$

Moreover, we have that

$$\frac{\partial}{\partial t} v(t, y_0) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} a_k \cos\left(\sqrt{\lambda_k} t\right) \Phi_k(y_0)$$

since this series is also uniformly convergent on $[0, \infty) \times V$ for $u \in H_0^1(V)$. Therefore,

$$\lim_{t \downarrow 0} \frac{\partial}{\partial t} v(t, y_0) = \sum_{n=1}^{\infty} \sqrt{\lambda_k} a_k \; \Phi_k(y_0) = u(y_0).$$

■

CHAPTER 4

Non-linear elliptic equations on p.c.f. self-similar fractals

Let V be a p.c f. self-similar fractal in $\mathbb{R}^n$ that possesses a regular harmonic structure and satisfies the separation condition (1.17), and let V_0 be its boundary. Let μ be a regular Borel measure on V with $0 < \mu(U) < \infty$ for any non-empty open subset U of V. In this chapter we investigate nonlinear elliptic equations of the form

$$\Delta u + a(x)u = f(x,u), \quad x \in V \backslash V_0, \tag{4.1}$$

with zero Dirichlet boundary conditions

$$u|_{V_0} = 0. \tag{4.2}$$

We obtain a number of results on the existence of non-trivial solutions of (4.1), (4.2). We use the mountain pass theorem and the saddle point theorem to study such equations for different classes of a and f. The Sobolev-type inequality (1.13) leads to properties that contrast with those for classical domains. It is interesting to note that we do not need the growth restriction of $f(x,t)$ in t, which is essential in the classical existence theory of non-trivial weak solutions to (1.1) for domains in $\mathbb{R}^{n-1}, n \geq 4$, see [**1, 19, 43, 44**]. Also, the condition on $f(x,t)$ near $t = 0$ is considerably weaker than in the classical case, see the condition (f_1) in Section 4.2. The work here is motivated by the classical results and methods of [**1, 44**], but there are considerable differences that are a consequence of the Sobolev-type inequality (1.13) that holds in this fractal situation. The main results of this chapter appeared in [**12**]. Falconer [**11**] obtained existence results in the special case where $a(x) \equiv 0$ for more general fractal domains appealing to a Poincaré inequality assumption proposed by Mosco [**39**].

4.1. Variational principle.

We say $u \in H_0^1(V)$ is a *weak solution* to (4.1)-(4.2) if

$$\begin{aligned} W(u,v) &- \int_V a(x)u(x)v(x)d\mu(x) \\ &+ \int_V f(x,u(x))v(x)d\mu(x) = 0 \quad \text{for all } v \in H_0^1(V). \end{aligned} \tag{4.3}$$

Moreover $u \in H_0^1(V)$ is said to be a *strong solution* to (4.1)-(4.2) if (4.1) holds for all points $x \in V \backslash V_0$, where Δ is understood to be the standard Laplacian given in (1.27).

Let E be a real Banach space with a norm $\| \ \|$. Recall that a functional $I : E \to \mathbb{R}$ is *Fréchet differentiable* at $u \in E$ if there exists a continuous linear map $I'(u) : E \to \mathbb{R}$ satisfying

$$\lim_{\|v\| \to 0} \frac{|I(u+v) - I(u) - I'(u)v|}{\|v\|} = 0.$$

The mapping $I'(u)$ is termed the Fréchet derivative of I at u. We say $u \in E$ is a *critical point* of a functional I if $I'(u) = 0$. A real number c is said to be a *critical value* of I if the set $\{u \in E | I(u) = c \text{ and } I'(u) = 0\}$ is not empty. Let $C^1(E, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable with continuous Fréchet derivatives on E.

We now define $I : H_0^1(V) \to \mathbb{R}$ by

$$I(u) = \frac{1}{2} W(u,u) - \frac{1}{2} \int_V a(x) u(x)^2 d\mu(x) + \int_V F(x, u(x)) d\mu(x), \tag{4.4}$$

where

$$F(x,u) = \int_0^u f(x,t) dt. \tag{4.5}$$

Proposition 4.1 (variational principle). *Suppose that $f(x,t) \in C(V \times \mathbb{R}, \mathbb{R})$ and $\int_V |a(x)| d\mu(x) < \infty$. Then the functional I given by (4.4) belongs to $C^1(H_0^1(V), \mathbb{R})$. Moreover,*

$$\begin{aligned} I'(u)v = W(u,v) &- \int_V a(x) u(x) v(x) d\mu(x) \\ &+ \int_V f(x, u(x)) v(x) d\mu(x), \qquad v \in H_0^1(V), \end{aligned} \tag{4.6}$$

for each point $u \in H_0^1(V)$. In particular, u is a weak solution of (4.1)-(4.2) if and only if $I'(u) = 0$.

Proof. For $u \in H_0^1(V)$, let

$$L(u) = \int_V F(x, u(x)) d\mu(x). \tag{4.7}$$

We must show that

$$L'(u)v = \int_V f(x, u(x)) v(x) d\mu(x), \qquad v \in H_0^1(V), \tag{4.8}$$

for each $u \in H_0^1(V)$. For $u, v \in H_0^1(V)$, by (1.22),

$$\begin{aligned}\left|L(u+v) - L(u) - \int_V f(x,u)v d\mu\right| &= \left|\int_V \left(\int_u^{u+v} f(x,t)dt - f(x,u)v\right) d\mu\right| \\ &\leq \int_V |f(x,u+\xi v) - f(x,u)| \; |v| d\mu \\ &\leq M\|v\| \int_V |f(x,u+\xi v) - f(x,u)| d\mu,\end{aligned}$$

where $0 \leq \xi \leq 1$. Since $f(x,t)$ is continuous on $V \times \mathbb{R}$, it follows that

$$\begin{aligned}&\lim_{\|v\|\to 0} \frac{|L(u+v) - L(u) - \int_V f(x,u)v d\mu|}{\|v\|} \\ &\leq M \lim_{\|v\|\to 0} \int_V |f(x,u+\xi v) - f(x,u)| d\mu \\ &= 0\end{aligned}$$

and thus $L(u)$ has the Fréchet derivative given by (4.8) at each point $u \in H_0^1(V)$. From (4.8) and (1.22), it is easy to see that $L'(u)$ is continuous on $H_0^1(V)$ since $f(x,t) \in C(V \times \mathbb{R}, \mathbb{R})$. Similarly, one can easily show that the functional

$$\frac{1}{2}\int_V a(x)u^2 d\mu \in C^1(H_0^1(V), \mathbb{R})$$

provided that $\int_V |a(x)| d\mu < \infty$, and its Fréchet derivative is the second term in the right side in (4.6). It is clear that the functional $\frac{1}{2}W(u,u) \in C^1(H_0^1(V), \mathbb{R})$ and its Fréchet derivative is given by $W(u,v)$. The last remark follows from (4.3). ∎

Remark 4.2. The above argument shows that $I \in C^1(H_0^1(V), \mathbb{R})$ provided that f is continuous and a is integrable. No growth conditions are placed on the nonlinear function f.

A useful technical assumption - the Palais-Smale condition - repeatedly occurs in critical point theory. For $I \in C^1(E, \mathbb{R})$, we say I satisfies the *Palais-Smale condition* (henceforth denoted by (PS)) if

(PS) *every sequence* $\{u_k\} \subset E$ *for which* $I(u_k)$ *is bounded and* $I'(u_k) \to 0$ *as* $k \to \infty$ *possesses a convergent subsequence.*

We will see below that in the case of p.c.f. fractals (PS) can be obtained for I without recourse to growth conditions of $f(x,t)$ in t. This is very different from classical domains, where $|f(x,t)|$ is required to grow slower than $|t|^s, 1 < s < \frac{n+2}{n-2}$ if $n \geq 4$ and slower than $\exp(t^2)$ if $n = 2$, for domains in $\mathbb{R}^{n-1}$, so that Sobolev's embedding theorem can be applied. See **[1, 19, 43, 44]**.

The following proposition simplifies checking of the condition (PS) given the Sobolev-type inequality, compare **[43, 44]**.

Proposition 4.3. *Let I be given by (4.4). If the sequence $\{u_k\}$ is bounded in $H_0^1(V)$ and $I'(u_k) \to 0$, then $\{u_k\}$ has a convergent subsequence in $H_0^1(V)$.*

Proof. If $\{u_k\}$ is bounded in $H_0^1(V)$, by (1.21) there exist a subsequence, which we denote by $\{u_k\}$ for the remainder of this proof, and a function $u \in C_0(V)$ such that $u_k \to u$ as $n \to \infty$ in $C_0(V)$. From (4.6) and (1.22), it follows that

$$\begin{aligned}
\|u_l - u_j\| &= \sup_{\|v\|\leq 1} |W(u_l - u_j, v)| \\
&= \sup_{\|v\|\leq 1} \Big|I'(u_l)v - I'(u_j)v + \int_V a(x)(u_l - u_j)v d\mu \\
&\quad - \int_V (f(x,u_l) - f(x,u_j))v d\mu\Big| \\
&\leq \|I'(u_l)\| + \|I'(u_j)\| + M \max_{x\in V}|u_l(x) - u_j(x)| \int_V |a(x)| d\mu \\
&\quad + M\int_V |f(x,u_l) - f(x,u_j)| d\mu \\
&\to 0, \qquad \text{as } \ l,j \to \infty,
\end{aligned}$$

since $I'(u_k) \to 0$ and using the Lebesgue dominated convergence theorem. ∎

Proposition 1.7 gives rise to regularity of weak solutions to (4.1)-(4.2) for continuous functions $a(x)$ and $f(x,t)$.

Lemma 4.4 (regularity). *Assume that $u \in H_0^1(V)$ is a weak solution to (4.1)-(4.2) with $a(x) \in C(V)$ and $f(x,t) \in C(V \times \mathbb{R})$. Then u is a strong solution to (4.1)-(4.2).*

Proof. Since $a \in C(V)$, $f \in C(V \times \mathbb{R})$ and $u \in H_0^1(V)$, the functions $a(x)u(x)$ and $f(x,u(x))$ are continuous on V. It follows from (1.23) and (4.3) that

$$\Delta u = f(x,u) - a(x)u \qquad \text{in } L^2(V),$$

so Δu is continuous on V, giving that $\Delta u = \Delta_s u$ in $V\backslash V_0$ by Proposition 1.7. ∎

Lemma 4.4, together with the existence theorems for weak solutions of (4.1) and (4.2), see below, implies that for suitable continuous functions a and f the solutions are also strong solutions.

4.2. Existence theorems using the mountain pass theorem

We recall the mountain pass theorem, see [1, 44]. Let E be a real Banach space and write $\overline{B}_\rho$ for the closed ball with radius ρ in E and $\partial B_\rho = \{u \in E : \|u\| = \rho\}$ is the boundary of $\overline{B}_\rho$.

Proposition 4.5 (mountain pass theorem). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfies (PS). Suppose that $I(0) = 0$ and*

(I_1) there are positive constants ρ and σ such that $I|_{\partial B_\rho} \geq \sigma$, and

(I_2) there exists $\zeta \in E \backslash \overline{B}_\rho$ such that $I(\zeta) \leq 0$.

Then I possesses a critical value $c \geq \sigma$, which can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) : \ g(0) = 0 \text{ and } g(1) = \zeta\}.$$

■

We now impose conditions (f_1) and (f_2) on f and (a_1) on a that ensure that (4.1)-(4.2) has a non-trivial solution. In order to apply Proposition 4.5 to I given by (4.4), we assume that

(f_1) there are positive constants M_0 and β such that

$$b(M_0) \equiv \max_{\substack{x \in V \\ |t| \leq M_0}} |f(x,t)| \leq \frac{M_0}{2(\beta+1)M^2},$$

(f_2) there are constants $\nu > 2$ and $r \geq 0$ such that for $|t| \geq r$ and all $x \in V$,

$$tf(x,t) \leq \nu F(x,t) < 0,$$

and

(a_1) $a \in L^1(V)$ and either $a(x) \leq 0$ for all $x \in V$ or $\int_V |a(x)| d\mu(x) < M^{-2}$, where M is the same as in (1.22).

Remarks 4.6.

(i). If

$$\lim_{t \to 0} \left| \frac{f(x,t)}{t} \right| = 0 \qquad \text{uniformly in } x \in V, \tag{4.9}$$

then (f_1) is satisfied. However (4.9) does not imply (f_1). For instance, if $f(x,t) = \sin t$ for $|t| \leq 4M^2$, and $M_0 = 4M^2$ and $\beta = 1$, then (f_1) is satisfied since $b(M_0) = 1$, but (4.9) is not true. Condition (4.9) is a usual assumption on f near zero in the classical setting for regular domains, see [1, 44].

(ii). Integrating the inequality of (f_2) shows that there exist positive constants b_1, b_2 such that

$$F(x,t) \leq -b_1|t|^{\nu} + b_2, \qquad \text{for all } x \in V \text{ and } t \in \mathbb{R}. \tag{4.10}$$

(iii). Let $f(x,t) = -t|t|^{s-1}, s > 1$. Then it is easy to see that $f(x,t)$ satisfies (f_1) and (f_2). This is the basic example of a function f satisfying the conditions of the next theorem. We shall see that such an $f(x,t)$ also satisfies the conditions (f_3) and (f_4) below.

Theorem 4.7. *If $f \in C(V \times \mathbb{R})$ satisfies (f_1) and (f_2) and a satisfies (a_1), then (4.1)-(4.2) possesses a non-trivial weak solution. Moreover, if a is continuous, then this solution is also a strong solution.*

Proof. The proof follows the pattern of the classical case, see [**1, 44**]. Let $E = H_0^1(V)$ and let I be defined by (4.4). From Proposition 4.1, $I \in C^1(E, \mathbb{R})$; the weak solution of (4.1) will be obtained as a critical point of I using Proposition 4.5. By (a_1) and (1.22), it follows that

$$\|u\| = \left(W(u,u) - \int_V a(x)u^2 d\mu\right)^{\frac{1}{2}}, \tag{4.11}$$

is an equivalent norm to $\sqrt{W(u,u)}$ on $H_0^1(V)$, a notation we adopt for the remainder of the proof. Thus in (4.4)

$$I(u) = \frac{1}{2}\|u\|^2 + \int_V F(x,u)d\mu. \tag{4.12}$$

We first verify that I satisfies (I_1). Taking M_0 as in (f_1), we see that (4.5) gives, using (1.22),

$$|F(x,u)| \leq \int_0^{|u|} |f(x,t)|dt \leq M_0 b(M_0).$$

Therefore, for $u \in E$ with $\|u\| = \rho = \frac{M_0}{M}$, we see that (4.12) and (f_1) give

$$\begin{aligned} I(u) &\geq \frac{1}{2}\rho^2 - M_0 b(M_0) \\ &\geq \frac{1}{2}\frac{M_0^2}{M^2} - \frac{M_0^2}{2(\beta+1)M^2} \\ &= \frac{\beta}{\beta+1} \cdot \frac{M_0^2}{2M^2}, \end{aligned} \tag{4.13}$$

and so (I_1) holds with $\rho = \frac{M_0}{M}$ and $\sigma = \frac{\beta}{\beta+1} \cdot \frac{M_0^2}{2M^2}$.

To verify (I_2), note that by (4.10),

$$J(u) = \int_V F(x,u)d\mu \leq -b_1 \int_V |u|^{\nu} d\mu + b_2, \tag{4.14}$$

for all $u \in E$. Choosing $u \in E$ with $\|u\| = 1$ and setting $N_1 = \int_V |u|^\nu d\mu > 0$,

$$\begin{aligned} I(su) &= \frac{1}{2}s^2 + \int_V F(x, su)d\mu \\ &\leq \frac{1}{2}s^2 - b_1 N_1 s^\nu + b_2 \\ &\to -\infty, \end{aligned} \tag{4.15}$$

as $s \to \infty$, since $\nu > 2$; thus (I_2) holds.

It remains to verify that I satisfies (PS). Suppose that $|I(u_m)| \leq b$ and $I'(u_m) \to 0$. We have for sufficiently large m, using (4.12),

$$\begin{aligned} b+\nu^{-1}\|u_m\| &\geq I(u_m) - \nu^{-1} I'(u_m)u_m \\ &= \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_m\|^2 + \nu^{-1}\int_V (\nu F(x, u_m) - f(x, u_m)u_m)d\mu \\ &= \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_m\|^2 + \nu^{-1}\int_{\{x\in V:\, |u_m|<r\}} (\nu F(x, u_m) - f(x, u_m)u_m)d\mu \\ &\quad + \nu^{-1}\int_{\{x\in V:\, |u_m|\geq r\}} (\nu F(x, u_m) - f(x, u_m)u_m)d\mu. \end{aligned} \tag{4.16}$$

By (f_2), the third term in (4.16) is non-negative while the second term is bounded by a constant independent of m. Since $\frac{1}{2} - \frac{1}{\nu} > 0$, (4.16) implies that $\{u_m\}$ is bounded in $H_0^1(V)$. Therefore, by Proposition 4.3, I satisfies (PS).

Finally, note that $I(0) = 0$, so by Proposition 4.5 that there is a critical point u with $I(u) \geq \sigma > 0$. Therefore, u is a non-trivial weak solution of (4.1) by Proposition 4.1. Lemma 4.4 then gives the strong solutions as stated. ■

Note that (1.22) plays a crucial rôle in the proof of Theorem 4.7 analogous to the Sobolev embedding theorems in the classical case, see [**1, 19, 44**].

Next we consider (4.1)-(4.2) when $a(x)$ is non-negative and large (for instance, not satisfying (a_1)). We now suppose that $a(x)$ is bounded, that is there is a constant $\rho_1 > 0$ such that

$$0 \leq a(x) \leq \rho_1. \tag{a_2}$$

We shall show that (4.1)-(4.2) has a non-trivial solution if β is large in a sense involving the least eigenvalue of Δ greater than 1.

From Chapter 2, we know that if $a(x) \geq 0$ with $0 < \int_V a(x)d\mu < \infty$, then the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda a(x)u &= 0, \\ u|_{V_0} &= 0, \end{aligned} \tag{4.17}$$

possesses a sequence of eigenvalues $\{\lambda_m\}$ with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \qquad \lambda_m \to \infty \text{ as } m \to \infty,$$

and the corresponding eigenfunctions $\{\psi_m\}$ form an orthogonal basis of the Hilbert space $H_0^1(V)$.

Let $u \in H_0^1(V)$. We may write u as

$$u = \sum_{k=1}^{\infty} a_k \psi_k, \qquad a_k \in \mathbb{R}, \quad k \geq 1. \tag{4.18}$$

If $\lambda_1 > 1$, we have

$$\begin{aligned} W(u,u) &= \sum_{k=1}^{\infty} a_k^2 W(\psi_k, \psi_k) \\ &= \sum_{k=1}^{\infty} a_k^2 \lambda_k \int_V a(x)\psi_k^2 d\mu \\ &\geq \lambda_1 \int_V a(x) u^2 d\mu, \end{aligned} \tag{4.19}$$

and so

$$W(u,u) - \int_V a(x)u^2 d\mu \geq \left(1 - \frac{1}{\lambda_1}\right) W(u,u).$$

Thus $(W(u,u) - \int_V a(x)u^2 d\mu)^{1/2}$ can again be taken as a norm on $H_0^1(V)$. Hence if f satisfies (f_1) and (f_2) and $a(x)$ is non-negative and integrable on V, and $\lambda_1 > 1$, one can use Proposition 4.5 to establish the existence result for a weak solution to (4.1) as in Theorem 4.7. However, if $\lambda_1 \leq 1$ and (I_1) no longer holds, the following variant of the mountain pass theorem is needed where a milder version of (I_1) is satisfied.

Proposition 4.8. *Let E be a real Banach space with $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$ satisfies (PS), and*

- (I_1'): *there are positive constants ρ and σ such that $I|_{\partial B_\rho \cap E_2} \geq \sigma$,*
- (I_3): *there is an $\zeta \in \partial B_1 \cap E_2$ and $R > \rho$ such that if $Q \equiv (\overline{B}_R \cap E_1) \oplus \{r\zeta | 0 < r < R\}$, then $I|_{\partial Q} \leq 0$, where ∂Q is the boundary of Q relative to $E_1 \oplus span\{\zeta\}$.*

Then I possesses a critical value $c \geq \sigma$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

where

$$\Gamma = \{h \in C(\overline{Q}, E) : \ h = id \ \ on \ \ \partial Q\}.$$

Proof. See [44]. ∎

Remark 4.9. Suppose $I|_{E_1} \le 0$ and there is a $\zeta \in \partial B_1 \cap E_2$ and an $\overline{R} > \rho$ such that $I(u) \le 0$ for $u \in E_1 \oplus \text{span}\{\zeta\}$ and $\|u\| \ge \overline{R}$. Then for large R, the set Q as defined in (I_3) satisfies $I|_{\partial Q} \le 0$, see [44].

Theorem 4.10. *Let λ_{p+1} be the smallest eigenvalue of (4.17) that is greater than 1. Suppose that a satisfies (a_2), and $f \in C(V \times \mathbb{R})$ satisfies (f_1) and (f_2) with $\beta > \frac{1}{\lambda_{p+1}-1}$ and*

(f_3) $\quad tf(x,t) \le 0$ *for all* $t \in \mathbb{R}$.

Then (4.1)-(4.2) possesses a non-trivial weak solution. Moreover, if a is continuous, then this solution is also a strong solution.

Proof. Let λ_1 be the smallest eigenvalue of (4.17). If $\lambda_1 > 1$, Theorem 4.10 follows from the above remarks before Proposition 4.8. Thus we suppose $\lambda_1 \le 1$ in what follows. We will show that I defined by (4.4) satisfies the assumptions of Proposition 4.8. Clearly $I \in C^1(E, \mathbb{R})$ by Proposition 4.1. Set $E_1 = \text{span}\{\psi_1, \psi_2, \cdots, \psi_p\}$ and $E_2 = E_1^{\perp}$. For $u \in E_2$ we get, as in (4.19),

$$W(u,u) - \int_V a(x)u^2 d\mu \ge \left(1 - \frac{1}{\lambda_{p+1}}\right) W(u,u),$$

while by (f_1)

$$\int_V F(x,u)d\mu \ge -M_0 b(M_0) \ge -\frac{M_0^2}{2(\beta+1)M^2}.$$

Therefore, for each $u \in E_2$ with $\sqrt{W(u,u)} = \|u\| = \frac{M_0}{M}$, using (4.4) and that $\beta > \frac{1}{\lambda_{p+1}-1}$,

$$\begin{aligned} I(u) &\ge \frac{1}{2}\left(1 - \frac{1}{\lambda_{p+1}}\right) \cdot \frac{M_0^2}{M^2} - \frac{M_0^2}{2(\beta+1)M^2} \\ &= \frac{1}{2}\left(1 - \frac{1}{\lambda_{p+1}} - \frac{1}{\beta+1}\right)\frac{M_0^2}{M^2} \\ &> 0. \end{aligned}$$

Hence I satisfies (I_1'). To verify (I_3), we first show that $I|_{E_1} \le 0$. For $u \in E_1$, we may write $u = \sum\limits_{k=1}^{p} a_k \psi_k$ $(a_k \in \mathbb{R})$, and since $\lambda_p \le 1$,

$$W(u,u) - \int_V a(x)u^2 d\mu = \sum_{k=1}^{p}\left(1 - \frac{1}{\lambda_k}\right) a_k^2 W(\psi_k, \psi_k) \le 0,$$

By (f_3), $F(x,t) \le 0$ for all $x \in V$ and $t \in \mathbb{R}$, so $I|_{E_1} \le 0$.

Next we assume without loss of generality that $\|\psi_j\| = 1$ for $1 \le j \le p+1$. Let $\zeta = \psi_{p+1}$ and $u = \sum\limits_{k=1}^{p+1} a_k \psi_k \in E_1 \oplus \text{span}\{\zeta\}$ with $\sqrt{W(u,u)} = \|u\| = 1$. For $s \in \mathbb{R}$, (4.4) and (4.10) give

$$
\begin{aligned}
I(su) &= \frac{1}{2}s^2\|u\|^2 - \frac{1}{2}s^2\int_V a(x)u^2 d\mu + \int_V F(x,su)d\mu \\
&= \frac{1}{2}s^2\sum_{k=1}^{p+1}\left(1-\frac{1}{\lambda_k}\right)a_k^2 + \int_V F(x,su)d\mu \\
&\le \frac{1}{2}s^2\left(1-\frac{1}{\lambda_{p+1}}\right)a_{p+1}^2 - b_1 s^\nu \int_V \left|\sum_{k=1}^{p+1} a_k\psi_k\right|^\nu d\mu + b_2 \\
&\le \frac{1}{2}s^2\left(1-\frac{1}{\lambda_{p+1}}\right) - b_3 s^\nu + b_2,
\end{aligned}
\tag{4.20}
$$

where b_3 is a positive constant independent of u and s, since $|a_{p+1}| \le 1$ and

$$
\int_V \left|\sum_{k=1}^{p+1} a_k\psi_k\right|^\nu d\mu \ge b_0 > 0
$$

uniformly for $(a_1, a_2, \cdots, a_{p+1}) \in \mathbb{R}^{p+1}$ with $\sum_{k=1}^{p+1} a_k^2 = 1$. Therefore, $I(u) \le 0$ for $u \in E_1 \oplus span\{\zeta\}$ with $\|u\|$ large enough. Then I satisfies (I_3) by Remark 4.9.

It remains to check that I satisfies (PS). By Proposition 4.3, it suffices to show that if $|I(u_k)| \le b$ and $I'(u_k) \to 0$ then $\{u_k\}$ is bounded in $H_0^1(V)$. Choose $\gamma \in (\frac{1}{\nu}, \frac{1}{2})$. Then for k large, by (f_2), (4.4), (4.6), (4.10) and (a_2),

$$
\begin{aligned}
b + \gamma\|u_k\| &\ge I(u_k) - \gamma I'(u_k)u_k = \left(\frac{1}{2}-\gamma\right)\|u_k\|^2 \\
&- \left(\frac{1}{2}-\gamma\right)\int_V a(x)u_k^2\, d\mu + \int_V \left(F(x,u_k) - \gamma f(x,u_k)u_k\right) d\mu \\
&\ge \left(\frac{1}{2}-\gamma\right)\|u_k\|^2 - \left(\frac{1}{2}-\gamma\right)\int_V a(x)u_k^2 d\mu - (\gamma\nu - 1)\int_V F(x,u_k)d\mu \\
&\ge \left(\frac{1}{2}-\gamma\right)\|u_k\|^2 - \left(\frac{1}{2}-\gamma\right)\int_V a(x)u_k^2 d\mu + b_1(\gamma\nu-1)\int_V |u_k|^\nu d\mu - b_4 \\
&\ge \left(\frac{1}{2}-\gamma\right)\|u_k\|^2 - \left(\frac{1}{2}-\gamma\right)\rho_1\int_V u_k^2 d\mu \\
&\quad + b_1(\gamma\nu-1)\int_V |u_k|^\nu d\mu - b_4,
\end{aligned}
\tag{4.21}
$$

where b_4 is a constant independent of k. Using Hölder's and Young's inequalities, we have

$$
\begin{aligned}
\int_V u_k^2 d\mu &\le \left(\int_V |u_k|^\nu d\mu\right)^{\frac{2}{\nu}} \\
&\le \epsilon\int_V |u_k|^\nu d\mu + \eta(\epsilon),
\end{aligned}
\tag{4.22}
$$

for all $\epsilon > 0$, where $\eta(\epsilon)$ is a constant depending only on ϵ and ν, but independent of k. Choosing ϵ so small that $b_1(\gamma\nu - 1) \ge (\frac{1}{2}-\gamma)\rho_1\epsilon$, it follows from (4.21)-(4.22) that

$$b + \gamma\|u_k\| \geq \left(\frac{1}{2} - \gamma\right)\|u_k\|^2 - b_5$$

and thus $\{u_k\}$ is bounded in $H_0^1(V)$. The stated result follows from Proposition 4.5, Proposition 4.1 and Lemma 4.4. ■

It is interesting to establish the existence of multiple solutions of (4.1)-(4.2). To do this, another version of the mountain pass theorem is needed, see [**1, 44**].

Proposition 4.11. *Let E be a real Banach and let $I \in C^1(E, \mathbb{R})$ be even (that is, $I(-u) = I(u)$ for all $u \in E$), satisfy (PS), and with $I(0) = 0$. If $E = E_1 \oplus E_2$, where E_1 is finite dimensional and I satisfies*

- (I_1'): *there are positive constants ρ and σ such that $I|_{\partial B_\rho \cap E_2} \geq \sigma$, and*
- (I_2'): *for each finite dimensional subspace $\tilde{E} \subset E$, there is an $\rho_2 = \rho_2(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \backslash B_{\rho_2}$.*

Then I possesses an unbounded sequence of critical values.■

We next assume that f satisfies in (4.1)-(4.2)

(f_4) $\quad f(x,t)$ is odd in t.

Theorem 4.12. *Suppose that $f \in C(V \times \mathbb{R})$ satisfies $(f_1), (f_2)$ and (f_4) and a satisfies (a_2). Then (4.1)-(4.2) possesses an unbounded sequence of weak solutions in Hilbert space $H_0^1(V)$. If a is continuous, then these solutions are strong solutions.*

Proof. With $E = H_0^1(V)$ as usual and

$$I(u) = \frac{1}{2}W(u,u) - \frac{1}{2}\int_V a(x)u^2 d\mu + \int_V F(x,u)d\mu,$$

Proposition 4.1 shows that $I(u) \in C^1(E, \mathbb{R})$, and $I(0) = 0$. As in the proof of Theorem 4.10, $I(u)$ satisfies (PS), and (f_4) implies I is even. The estimates of (4.20) show that I satisfies (I_2'). To verify (I_1'), let $\{\psi_j\}_1^\infty$ be a sequence of eigenfunctions of (4.17), let $E_1 = \text{span}\{\psi_1, \cdots, \psi_k\}$ where k is so large that

$$1 - \frac{1}{\lambda_k} - \frac{1}{\beta+1} \geq \frac{\beta}{2(\beta+1)},$$

and set $E_2 = E_1^\perp$.

For $u \in E_2$ with $\|u\| = \frac{M_0}{M}$, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_k}\right)\|u\|^2 + \int_V F(x,u)d\mu \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_k}\right)\|u\|^2 - \frac{M_0^2}{2(\beta+1)M^2} \\ &= \frac{1}{2}\left(1 - \frac{1}{\lambda_k} - \frac{1}{\beta+1}\right)\frac{M_0^2}{M^2} \\ &\geq \frac{\beta}{4(\beta+1)}\frac{M_0^2}{M^2}, \end{aligned}$$

giving (I_1') where $\rho = \frac{M_0}{M}$ and $\sigma = \frac{\beta}{4(\beta+1)} \cdot \frac{M_0^2}{M^2}$. Thus Proposition 4.11 implies that I possesses an unbounded sequence of critical values $c_j = I(u_j)$, where u_j is a weak solution of (4.1). Since $I'(u_j)u_j = 0$,

$$W(u_j,u_j) - \int_V a(x)u_j^2 d\mu + \int_V f(x,u_j)u_j d\mu = 0,$$

and it follows that

$$c_j = \int_V \left(F(x,u_j) - \frac{1}{2}f(x,u_j)u_j\right)d\mu \to \infty,$$

as $j \to \infty$. Therefore, by (1.21) and (1.22) $\{u_j\}$ is unbounded in $H_0^1(V)$. As before this leads to the solutions specified. ■

Remark 4.13. Note that the above existence theorems for non-trivial solutions to (4.1)-(4.2) do not depend on the Euclidean dimension $n-1$ of the space $\mathbb{R}^{n-1}$ in which the p.c.f. self-similar V lies. This is very different from the classical case, where non-trivial solutions on an open subset of $\mathbb{R}^{n-1}$ need not exist if $n \geq 4$ by Pohozaev's argument [41].

4.3. Existence theorems using the saddle point theorem

All the existence theorems above assume the condition (f_2). However, if $f(x,t)$ is constant for large $|t|$, the condition (f_2) is violated. But for such a case, (4.1) is linear for large $|t|$ and would be expected to have a non-trivial solution. The saddle point theorem copes with this case.

Proposition 4.14 (saddle point theorem). *Let $E = E_1 \oplus E_2$ be a real Banach space with $E_1 \neq \{0\}$ and finite dimensional. Suppose that $I \in C^1(E,\mathbb{R})$ satisfies (PS), and*

(I_4): *there is constant σ and a bounded neighborhood Ω of 0 in E_1 such that $I|_{\partial\Omega} \leq \sigma$, and*

(I_5): *there is a constant $\gamma > \sigma$ such that $I|_{E_2} \geq \gamma$.*

Then I possesses a critical value $c \geq \gamma$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{\Omega}} I(h(u)),$$

where

$$\Gamma = \{h \in C(\overline{\Omega}, E) : \ h = id \text{ on } \partial\Omega\}.$$

The proof of Proposition 4.14 is given in [**43, 44**]. ■

In (4.1)-(4.2) we now assume that f satisfies

(f_5): $f(x,t)$ is bounded in $V \times \mathbb{R}$, that is, there is a constant ρ_3 such that

$$|f(x,t)| \leq \rho_3 \qquad \text{for all} \quad (x,t) \in V \times \mathbb{R}, \qquad \text{and}$$

(f_6): $\quad F(x,\xi) = \int_0^\xi f(x,t)dt \to -\infty$, as $|\xi| \to \infty$ uniformly for $x \in V$,

and a satisfies

(a_3) $\quad a(x) \geq 0$ and $0 < \int_V a(x)d\mu < \infty$.

Theorem 4.15. *Assume that $f \in C(V \times \mathbb{R})$ satisfies (f_5) and (f_6) and a satisfies (a_3). Then (4.1)-(4.2) possesses a non-trivial solution. Moreover, if a is continuous, then this solution is also a strong solution.*

Proof. Let

$$I(u) = \frac{1}{2}W(u,u) - \frac{1}{2}\int_V a(x)u^2 d\mu + \int_V F(x,u)d\mu,$$

for all $u \in E \equiv H_0^1(V)$. Clearly $I \in C^1(E, \mathbb{R})$. It follows from (a_3) that (4.17) has a sequence of eigenvalues $\{\lambda_j\}_1^\infty$ in an increasing order with corresponding normalized eigenfunctions $\{\psi_j\}_1^\infty$ with $\|\psi_j\| = 1$ for all j. Let p be the integer such that $\lambda_p \leq 1 < \lambda_{p+1}$. We assume the most awkward case when $\lambda_p = 1 < \lambda_{p+1}$. If there is no such integer p, then the argument may be simplified by omitting the introduction of E^0.

Let $E_1 = \text{span}\{\psi_1, \cdots, \psi_p\}$, $E_2 = E_1^\perp$, and so $E = E_1 \oplus E_2$. We will show I satisfies (I_4), (I_5) and (PS) so that Theorem 4.15 follows from Proposition 4.14.

For $u \in E_2$, we write $u = \sum_{j=p+1}^{\infty} a_j\psi_j$, and

$$W(u,u) - \int_V a(x)u^2 d\mu \geq \left(1 - \frac{1}{\lambda_{p+1}}\right)\|u\|^2.$$

By (f_5) and (1.22)

$$|F(x,t)| \leq \int_0^{|u|} |f(x,t)|dt \leq M\rho_3\|u\|.$$

Therefore, for $u \in E_2$,

$$I(u) \geq \left(1 - \frac{1}{\lambda_{p+1}}\right)\|u\|^2 - M\rho_3\|u\|.$$

Thus I is bounded below on E_2 and so (I_5) holds.

Next if $u = \sum_{j=1}^{p} a_j\psi_j \in E_1$, then $u = u^0 + u^-$, where

$$u^0 \in E^0 = \operatorname{span}\{\psi_j|\ \lambda_j = 1\} \quad \text{and} \quad u^- \in E^- = \operatorname{span}\{\psi_j|\lambda_j < 1\}.$$

Thus

$$\begin{aligned} I(u) &= \frac{1}{2}\sum_{j<p} a_j^2\left(1 - \frac{1}{\lambda_j}\right) + \int_V F(x,u^0)d\mu \\ &\quad + \int_V \left(F(x,u^0+u^-) - F(x,u^0)\right)d\mu. \end{aligned} \tag{4.23}$$

By (f_5) and (1.22),

$$\left|\int_V \left(F(x,u^0+u^-) - F(x,u^0)\right)d\mu\right| \leq M\rho_3\|u^-\|, \tag{4.24}$$

and thus (4.23) gives

$$I(u) \leq -b_6\|u^-\|^2 + \int_V F(x,u^0)d\mu + M\rho_3\|u^-\|, \tag{4.25}$$

where $b_6 > 0$, unless E^- is empty in which case $\|u^-\| = 0$. We claim that

$$\int_V F(x,u^0)d\mu \to -\infty, \tag{4.26}$$

as $\|u^0\| \to \infty$ uniformly for $u^0 \in E^0$. Indeed, let $u^0 \in E^0$ and write $u^0 = t\psi$ where $\psi \in E^0$ with $\|\psi\| = 1$. Since $\psi(x) \not\equiv 0$ on V, there exists a subset $U = U(\psi)$ of V with $\mu(U) > 0$ such that $F(x,t\psi) \to -\infty$ on U by (f_6) as $t \to \infty$. From (f_5) and (f_6), we see that

$$\delta_0 \equiv \sup_{\substack{x\in V\\ \xi\in\mathbb{R}}} F(x,\xi) < \infty$$

and so

$$\begin{aligned} \int_V F(x,t\psi)d\mu &= \int_U F(x,t\psi)d\mu + \int_{V\setminus U} F(x,t\psi)d\mu \\ &\leq \int_U F(x,t\psi)d\mu + \delta_0 \\ &\to -\infty \qquad \text{as } t \to \infty. \end{aligned}$$

Note that the set $\{\psi : \psi \in E^0 \text{ and } \|\psi\| = 1\}$ is compact and the functional $\int_V F(x,u)d\mu \in C^1(E,\mathbb{R})$, so the claim (4.26) follows immediately. Since $\|u\|^2 =$

$\|u^-\|^2 + \|u^0\|^2$, we have that either $\|u^-\| \to \infty$ or $\|u^0\| \to \infty$ if $\|u\| \to \infty$. In each case we have that $I(u) \to -\infty$ as $\|u\| \to \infty$ by (4.25) and (4.26), and thus I satisfies (I_4).

Finally, we verify (PS). By Proposition 4.3, we need only show that $|I(u_k)| \leq b$ and $I'(u_k) \to 0$ implies $\{u_k\}$ is bounded in $H_0^1(V)$. We write $u_k = u_k^0 + u_k^- + u_k^+$, where $u_k^0 \in E^0, u_k^- \in E^-$ and $u_k^+ \in E_2$. We first show $\{u_k^\pm\}$ is bounded in $H_0^1(V)$. Since $I'(u_k) \to 0$ and $u_k^+ \in E_2$, for k sufficiently large,

$$\begin{aligned}
\|u_k^+\| &\geq \left|I'(u_k)u_k^+\right| \\
&= \Bigg| W(u_k^0 + u_k^- + u_k^+, u_k^+) - \int_V a(x)(u_k^0 + u_k^- + u_k^+)u_k^+ d\mu \\
&\quad + \int_V f(x, u_k)u_k^+ d\mu \Bigg| \\
&= \left| \|u_k^+\|^2 - \int_V a(x)(u_k^+)^2 d\mu + \int_V f(x, u_k)u_k^+ d\mu \right| \\
&\geq \left(1 - \frac{1}{\lambda_{p+1}}\right) \|u_k^+\|^2 - b_7\|u_k^+\|,
\end{aligned}$$

by virtue of (f_5) and (1.22), where b_7 is a constant independent of k, and thus $\{u_k^+\}$ is bounded in $H_0^1(V)$. In a similar way, $\{u_k^-\}$ is bounded in $H_0^1(V)$. It remains to show that $\{u_k^0\}$ is also bounded in $H_0^1(V)$. Observe that

$$\begin{aligned}
b &\geq \left|I(u_k)\right| \\
&= \Bigg| \frac{1}{2}\left(W(u_k^+, u_k^+) + W(u_k^-, u_k^-)\right) - \frac{1}{2}\int_V a(x)\left((u_k^+)^2 + (u_k^-)^2\right) d\mu \\
&\quad + \int_V (F(x, u_k) - F(x, u_k^0)) d\mu + \int_V F(x, u_k^0) d\mu \Bigg|.
\end{aligned}$$

Since $\{u_k^\pm\}$ is bounded in $H_0^1(V)$, it follows from (4.24) that there is a positive constant b_8 independent of k such that

$$\left| \int_V F(x, u_k^0) d\mu \right| \leq b_8,$$

and thus by (4.26) and (f_6), $\{u_k^0\}$ is also bounded in $H_0^1(V)$. Thus (PS) is satisfied, so the results follows from Proposition 4.14, Proposition 4.1 and Lemma 2.16. ■

CHAPTER 5

Nonlinear wave equations on p.c.f. self-similar fractals

Let V be a p.c.f. self-similar fractal in $\mathbb{R}^n$ $(n \geq 2)$ and V_0 its boundary. Let $\dim_H(V) = d_f$ and let μ be the normalized d_f-dimensional Hausdorff measure of V with $\mu(V) = 1$ as before. Assume that the *Sobolev-type inequality* (1.13) holds on V; by Theorem 1.3 this will certainly be the case if V satisfies the separation condition (1.17). We further assume that the *spectral dimension* d_s of V exists and $d_s < 2$; this is true if V is a p.c.f. self-similar fractal with a regular harmonic structure [**28**]. The nonlinear wave equation

$$u_{tt} = \Delta u + f(u), \quad x \in V \quad \text{and} \quad t > 0$$

with given initial data and zero boundary conditions is investigated in this chapter. We demonstrate the global existence of strong solutions for suitable f if the initial data belong to $E_{\theta-1/2}(V)$ where θ depends on the spectral dimension. We use the *wave propagator* and Hilbert spaces that we have constructed in Chapter 3. The main difficulty in obtaining the global existence of a solution is establishing a priori estimates depending on a regularity property for f. The regularity property of solutions is obtained through a fine analysis in which the Sobolev-type inequality plays a crucial rôle. Finally, in the opposite direction, we demonstrate the blow-up of strong solutions for certain functions f.

5.1. General Statements

We consider the nonlinear wave equation

$$u_{tt} = \Delta u + f(u), \quad t > 0,\ x \in V\backslash V_0, \tag{5.1}$$

with given initial data and boundary conditions

$$\begin{aligned} u|_{t=0} &= \phi(x), \quad x \in V, \\ u_t|_{t=0} &= \psi(x), \quad x \in V, \\ u|_{V_0} &= 0, \quad t > 0, \end{aligned} \tag{5.2}$$

where Δ is the weak Laplacian defined as in (1.23). The initial data ϕ and ψ are assumed to satisfy the compatibility condition $\phi|_{V_0} = \psi|_{V_0} = 0$. We suppose that $f(0) = 0$ and $f \in C^1(\mathbb{R})$ with f' locally Lipschitz continuous. We shall prove that (5.1), (5.2) possesses a global solution u in $H_0^1(V)$ for "dissipative" f if the initial data $\phi \in E_\theta(V)$ and $\psi \in E_{\theta-1/2}(V)$ with $\theta > (6+d_s)/4$. An example of such an f is $f(r) = -r|r|^p - m_0 r$ with $m_0 \geq 0$ and $p \geq 1$. The solution u is shown to have the property that u_{tt} and Δu exist on V for almost $t \in (0,\infty)$; thus (5.1) is interpreted as an equality on $(V\backslash V_0)$ for almost $t \in (0,\infty)$, see Section 5.2. In Section 5.3, we shall use the method initiated by Levine [**33**] to investigate the blow-up of *strong* solutions for f in a different class of functions, that is the strong solution tends to infinity as $t \to T$ for a finite time $T > 0$. An example of such an f is that $f(r) = r|r|^{p-1} - m_0 r$, where $p \geq 1$ and $m_0 \geq 0$.

The initial-boundary value problem (5.1), (5.2) was extensively investigated on bounded Euclidean domains with 'nice' boundary, see for example [**40, 47**]. Our approach for the fractal situation here is quite different. It is interesting to note that the restriction on the nonlinear function f is significantly weaker than in the non-fractal situation [**40, 47**].

Let $u, v \in C(V)$. From (1.5) in Chapter 1, we see that, using Minkowski's inequality,

$$W_0(uv,uv)^{1/2} \leq \|u\|_\infty W_0(v,v)^{1/2} + \|v\|_\infty W_0(u,u)^{1/2}, \tag{5.3}$$

where $\|u\|_\infty = \max\limits_{x\in V} |u(x)|$. By (1.7) we have that

$$W_m(uv,uv) = \sum_{\omega\in S^m} r_\omega\, W_0((u\circ F_\omega)(v\circ F_\omega),(u\circ F_\omega)(v\circ F_\omega)), \quad m \geq 1,$$

where $r_\omega = r_{i_1} r_{i_2}\cdots r_{i_m}$ for $\omega = i_1 i_2 \cdots i_m$ with $i_k \in S$ $(1 \leq k \leq m)$. Thus it follows from (5.3) that

$$\begin{aligned} W_m(uv,uv) &\leq \sum_{\omega\in S^m} r_\omega \Big\{ \|u\circ F_\omega\|_\infty W_0(v\circ F_\omega, v\circ \omega_w)^{1/2} \\ &\qquad + \|v\circ F_\omega\|_\infty W_0(u\circ F_\omega, u\circ F_\omega)^{1/2}\Big\}^2 \\ &\leq \sum_{\omega\in S^m} r_\omega \Big\{ \|u\|_\infty W_0(v\circ F_\omega, v\circ F_\omega)^{1/2} \\ &\qquad + \|v\|_\infty W_0(u\circ F_\omega, u\circ F_\omega)^{1/2}\Big\}^2 \quad (m \geq 1), \end{aligned}$$

Using Minkowski's inequality again, it follows that

$$\begin{aligned} W_m(uv,uv)^{1/2} \leq \Big\{ \sum_{\omega\in S^m} r_\omega \Big(&\|u\|_\infty W_0(v\circ F_\omega, v\circ F_\omega)^{1/2} \\ &+ \|v\|_\infty W_0(u\circ F_\omega, u\circ F_\omega)^{1/2}\Big)^2\Big\}^{1/2} \end{aligned}$$

$$\leq \|u\|_\infty \left\{ \sum_{\omega \in S^m} r_\omega W_0(v \circ F_\omega, v \circ F_\omega) \right\}^{1/2}$$

$$+ \|v\|_\infty \left\{ \sum_{\omega \in S^m} r_\omega W_0(u \circ F_\omega, u \circ F_\omega) \right\}^{1/2},$$

for all $m \geq 1$ and all $u, v \in C(V)$. Letting $m \to \infty$, we see that W satisfies

$$W(uv, uv)^{1/2} \leq \|u\|_\infty W(v,v)^{1/2} + \|v\|_\infty W(u,u)^{1/2} \tag{5.4}$$

for all $u, v \in C(V)$. Clearly W has a *strong Markovian property*

$$W(h(u), h(u)) \leq W(u,u) \quad \text{for all } u : V \to \mathbb{R}, \tag{5.5}$$

for all $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$|h(t_2) - h(t_1)| \leq |t_2 - t_1|, \quad t_1, t_2 \in \mathbb{R}.$$

5.2. Global existence of solutions

In this section we show the global existence of solutions to (5.1), (5.2) if the spectral dimension $d_s < 2$ and the initial data $\phi \in E_\theta(V)$ and $\psi \in E_{\theta - \frac{1}{2}}$, where $\theta > (6 + d_s)/4$. We employ the standard contraction principle to obtain the local existence of solutions which can be extended globally by using a priori estimates. We obtain a priori estimates for certain class of functions f.

Let

$$\Phi_k(x) = \frac{\varphi_k(x)}{\sqrt{\lambda_k}},$$

where $\{\varphi_k\}$ is the sequence of eigenfunctions of (3.2) corresponding to the eigenvalues $\{\lambda_k\}$. By a global *weak solution* of (5.1), (5.2) we mean $u(t) \equiv u(t,x) \in H_0^1(V)$ satisfying

$$u(t) = u_0(t) + \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \int_0^t \sin\left(\sqrt{\lambda_k}(t-\tau)\right) \, (f(u(\tau)), \Phi_k) \, d\tau, \tag{5.6}$$

for $t \geq 0$ and $x \in V$, where $(\ ,\ )$ is the inner product of $H_0^1(V)$ and

$$u_0(t) = \sum_{k=1}^{\infty} \frac{\sin\sqrt{\lambda_k} t}{\sqrt{\lambda_k}} \, b_k \Phi_k + \sum_{k=1}^{\infty} a_k \cos\left(\sqrt{\lambda_k} t\right) \, \Phi_k, \tag{5.7}$$

for $\phi(x) = \sum_{k=1}^{\infty} a_k \Phi_k(x) \in H_0^1(V)$ and $\psi(x) = \sum_{k=1}^{\infty} b_k \Phi_k(x) \in L^2(V)$. Observe that $u_0(t)$ is the solution of the linear wave equation (5.1), (5.2) with $f \equiv 0$, and has nice properties. However, the second term on the right-hand side in (5.6) is awkward in that the nonlinear function f takes effect, and so needs more analysis. Note that if $u(\tau) \in H_0^1(V)$ for $\tau \in (0,t)$, then $f(u(\tau)) \in H_0^1(V)$ for $\tau \in (0,t)$ by the strong

Markovian property (5.5), and so $(f(u(\tau)), \Phi_k)$ is defined. This is why we use the inner product of $H_0^1(V)$ in (5.6) rather than that of $E_\theta(V)$ for some $\theta > 1/2$.

5.2.1. Local Existence. In this subsection we establish the local existence of a weak solution of (5.1), (5.2), that is there exists a $u(t) \in H_0^1(V)$ satisfying (5.6) for $t \in (0, t_1)$, for some $t_1 > 0$.

Lemma 5.1 (local existence). *Let $d_s < 2$ and suppose the initial data $\phi = \sum_{k=1}^{\infty} a_k \Phi_k \in H_0^1(V)$ and $\psi = \sum_{k=1}^{\infty} b_k \Phi_k \in L^2(V)$. Writing $M_1 = \|\phi\| + \|\psi\|_2$, then there exists $t_1 > 0$ such that (5.1), (5.2) possesses a local weak solution $u(t) \in H_0^1(V)$ on $(0, t_1)$ satisfying*

$$\|u(t)\| \leq 2M_1, \quad t \in (0, t_1). \tag{5.8}$$

Proof. We use the standard contraction principle approach.

Let $E = \{u(t) \in H_0^1(V) : \|u(t)\| \leq 2M_1 \text{ for } t \in (0, t_1)\}$, where $t_1 > 0$ is to be determined below. Then E is a Banach space under the norm $\sup_{t\in(0,t_1)} \|u(t)\|$. Define a mapping $\mathcal{F}$ on E by

$$(\mathcal{F}u)(t) = u_0(t) + \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \int_0^t \sin\left(\sqrt{\lambda_k}(t - \tau)\right) \ (f(u(\tau)), \Phi_k)\, d\tau. \tag{5.9}$$

We show that $\mathcal{F}$ is a contracting mapping from E to E for all $t \in (0, t_1)$ for some small $t_1 > 0$. To see this, let $u(t) \in E$. Then $\|u(t)\| \leq 2M_1$ on $(0, t_1)$. We see from (1.22) that

$$\|u(t)\|_\infty \equiv \sup_{x\in V} |u(t, x)| \leq M \|u(t)\| \leq 2M\, M_1, \tag{5.10}$$

for all $(0, t_1)$. Since $\phi = \sum_{k=1}^{\infty} a_k \Phi_k \in H_0^1(V)$ and $\psi = \sum_{k=1}^{\infty} b_k \Phi_k \in L^2(V)$, we have from (5.7) that $u_0(t) \in E_{1/2}(V) = H_0^1(V)$ for all $t \geq 0$, since

$$\begin{aligned}
\|u_0(t)\| &= \left\{ \sum_{k=1}^{\infty} \left(\frac{\sin\sqrt{\lambda_k} t}{\sqrt{\lambda_k}} b_k + a_k \cos(\sqrt{\lambda_k} t) \right)^2 \right\}^{1/2} \\
&\leq \left\{ \sum_{k=1}^{\infty} \left(\frac{|b_k|}{\sqrt{\lambda_k}} + |a_k| \right)^2 \right\}^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{|b_k|^2}{\lambda_k} \right)^{1/2} + \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \\
&= \|\psi\|_2 + \|\phi\|,
\end{aligned} \tag{5.11}$$

for all $t \geq 0$.

Let

$$z(t) = \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \int_0^t \sin\left(\sqrt{\lambda_k}(t-\tau)\right) \, (f(u(\tau)), \Phi_k) \, d\tau. \tag{5.12}$$

It follows that, using Hölder's inequality,

$$\begin{aligned} \|z(t)\|^2 &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left(\int_0^t \sin\left(\sqrt{\lambda_k}(t-\tau)\right) \, (f(u(\tau)), \Phi_k) \, d\tau \right)^2 \\ &\le \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left(\int_0^t |(f(u(\tau)), \Phi_k)| \, d\tau \right)^2 \\ &\le \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left(t \int_0^t |(f(u(\tau)), \Phi_k)|^2 \, d\tau \right). \end{aligned} \tag{5.13}$$

Note that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} (g(u), \Phi_k)^2 = \|g(u)\|_2^2 \quad \text{for } g(u) \in L^2(V), \tag{5.14}$$

and

$$\|u\|_2 \le \frac{1}{\sqrt{\lambda_1}} \|u\| \quad \text{for all } u \in E_{1/2}(V), \tag{5.15}$$

where λ_1 is the first eigenvalue of (3.2). Therefore, setting $L = \sup\limits_{|r| \le 2M\,M_1} |f'(r)|$, we have from (5.13) that,

$$\begin{aligned} \|z(t)\|^2 &\le t \int_0^t \|f(u(\tau))\|_2^2 \, d\tau \\ &\le t \int_0^t L^2 \|u(\tau)\|_2^2 \, d\tau \\ &\le \frac{(2LM_1t)^2}{\lambda_1}, \end{aligned}$$

which implies that

$$\|z(t)\| \le \frac{2LM_1t_1}{\sqrt{\lambda_1}}, \quad \text{for all } t \in (0, t_1). \tag{5.16}$$

Thus $z(t) \in E_{1/2}(V) = H_0^1(V)$ for all $t \in (0, t_1)$ if $u(t) \in E$. Therefore, by (5.9) $(\mathcal{F}u)(t) \in H_0^1(V)$ for all $t \in (0, t_1)$ if $u(t) \in E$.

Combining (5.11) and (5.16), we have from (5.9) that for $u(t) \in E$,

$$\begin{aligned} \|(\mathcal{F}u)(t)\| &\le \|u_0(t)\| + \|z(t)\| \\ &\le \|\phi\| + \|\Phi\|_2 + \frac{2LM_1t_1}{\sqrt{\lambda_1}} \\ &= M_1 + \frac{2LM_1t_1}{\sqrt{\lambda_1}} \le 2M_1, \quad \text{for all } t \in (0, t_1), \end{aligned}$$

if we take $t_1 > 0$ such that $2Lt_1 \le \sqrt{\lambda_1}$. Hence, $\mathcal{F}$ maps E onto E for $t_1 \le \sqrt{\lambda_1}/(2L)$.

On the other hand, we get from (5.9) that for $u(t), v(t) \in E$,

$$\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \le \frac{Lt_1}{\sqrt{\lambda_1}} \sup_{\tau\in(0,t_1)} \|u(\tau) - v(\tau)\|$$

for all $t \in (0, t_1)$ in a similar way to the derivation of (5.16). Therefore,

$$\sup_{\tau\in(0,t_1)} \|(\mathcal{F}u)(\tau) - (\mathcal{F}v)(\tau)\| \le \frac{Lt_1}{\sqrt{\lambda_1}} \sup_{\tau\in(0,t_1)} \|u(\tau) - v(\tau)\|.$$

and so $\mathcal{F}$ is a contraction on E if we take $t_1 < \sqrt{\lambda_1}/L$. Thus the contraction principle guarantees the local existence of a weak solution $u(t) \in E$ to (5.1), (5.2), where $t_1 = \frac{\sqrt{\lambda_1}}{2L}$ with $L = \sup_{|r|\le 2MM_1} |f'(r)|$. ■

5.2.2. Regularity. In order to extend the local weak solution to $t \in (0, \infty)$, we require some a priori estimates, which may be obtained by using the regularity property of a local weak solution. In this subsection we prove that the local solution u obtained in Subsection 5.2.1 is *smooth* in t for certain initial data. For this, we use the concept of the *derivative* of a function in the norm $\| \ \|_2$.

Definition 5.2. *We say that $u(t) \in H_0^1(V)$ is differentiable at point $t_0 > 0$ in the norm $\| \ \|_2$ if there exists an element $\partial u(t_0)/\partial t \in H_0^1(V)$ such that*

$$\lim_{\delta\to 0} \left\| \frac{u(t_0+\delta) - u(t_0)}{\delta} - \frac{\partial u}{\partial t}(t_0) \right\|_2 = 0. \tag{5.17}$$

Without ambiguity, we denote the derivative of $u(t)$ at t_0 in the norm $\| \ \|_2$ by $\frac{\partial u}{\partial t}(t_0)$.

Proposition 5.3. *Assume that $u(t) \in H_0^1(V)$ is uniformly Lipschitz continuous in the norm $\| \ \|$ on $(0, t_1)$ for $t_1 > 0$, that is there exists some $M_2 > 0$ such that*

$$\|u(t+\delta) - u(t)\| \le M_2\delta \tag{5.18}$$

for all small $\delta > 0$ and $t \in (0, t_1 - \delta)$. Then $u(t)$ is differentiable in the norm $\| \ \|_2$ for almost all $t \in (0, t_1)$ (in the sense of Lebesgue measure), and for almost all $t \in (0, t_1)$,

$$\frac{\partial u(t)}{\partial t} = \sum_{k=1}^{\infty} a_k'(t)\,\varphi_k \in H_0^1(V), \tag{5.19}$$

where $u(t) = \sum\limits_{k=1}^{\infty} a_k(t)\varphi_k$ with $a_k(t) = \int_V u(t,x)\varphi_k(x)\, d\mu(x)$, and $a_k'(t)$ the usual derivative of $a_k(t)$ that exists almost everywhere on $(0, t_1)$.

Proof. From (5.18), we see that

$$\sum_{k=1}^{\infty} (a_k(t+\delta) - a_k(t))^2 \lambda_k = \|u(t+\delta) - u(t)\|^2 \le (M_2\delta)^2,$$

which gives

$$\sup_{k\geq 1} |a_k(t+\delta) - a_k(t)| \leq \sup_{k\geq 1}\left(\frac{M_2\delta}{\sqrt{\lambda_k}}\right) \leq \frac{M_2}{\sqrt{\lambda_1}}\delta, \quad \text{for all } t \in (0,t_1).$$

Thus for all $k \geq 1$, we see that $a_k(t)$ is uniformly Lipschitz continuous in the usual sense on $(0,t_1)$, and so $a'_k(t)$ exists almost everywhere on $(0,t_1)$. Let $z_0(t) = \sum_{k=1}^{\infty} a'_k(t)\varphi_k$ for those $t \in (0,t_1)$ for which $a'_k(t)$ exists for all $k \geq 1$. It is easy to see from (5.18) that

$$\|z_0(t)\|^2 = \sum_{k=1}^{\infty} a'_k(t)^2\lambda_k \leq M_2^2 \quad \text{for almost all } t \in (0,t_1),$$

so $z_0(t) \in H_0^1(V)$ for almost all $t \in (0,t_1)$. For fixed $t_0 \in (0,t_1)$ for which $a'_k(t_0)$ exists for all k, let

$$s_k(\delta) = \left(\frac{a_k(t_0+\delta) - a_k(t_0)}{\delta} - a'_k(t_0)\right)^2 \lambda_k.$$

Clearly

$$\sum_{k=1}^{\infty} s_k(\delta) \leq 2\,M_2^2,$$

for all small $\delta > 0$. Note that $1/\lambda_k^{1-2\theta}$ is decreasing in k and $1/\lambda_k^{1-2\theta} \to 0$ as $k \to \infty$, provided that $\theta < 1/2$, so using Dirichlet's criterion, we see that the series

$$\sum_{k=1}^{\infty} s_k(\delta)\,\frac{1}{\lambda_k^{1-2\theta}}$$

is uniformly convergent for small $\delta > 0$. Therefore, for $\theta < 1/2$,

$$\left\|\left|\frac{u(t_0+\delta) - u(t_0)}{\delta} - z_0(t_0)\right\|\right|_\theta^2 = \sum_{k=1}^{\infty} s_k(\delta)\,\frac{1}{\lambda_k^{1-2\theta}} \to 0 \quad \text{as } \delta \to 0.$$

In particular, taking $\theta = 0$ gives (5.17) with $z_0(t_0) = \frac{\partial u}{\partial t}(t_0)$ as required for (5.19). ∎

Proposition 5.4. *Let $z(t)$ be defined by (5.12). Assume that $u(t) \in H_0^1(V)$ for all $t \in (0,t_1)$. Then for all small $\delta > 0$ and all $t \in (0,t_1-\delta)$,*

$$\|z(t+\delta) - z(t)\| \leq M_3\delta + M_4\int_0^t \|u(\tau+\delta) - u(t)\|\,d\tau, \tag{5.20}$$

where $M_3, M_4 > 0$. Suppose further that $u(t)$ is Lipschitz continuous in $\|\ \|$ uniformly on $(0,t_1)$, that is there exists some $M_5 > 0$ such that

$$\|u(t+\delta) - u(t)\| \leq M_5\delta \quad \textit{for all } t \in (0,t_1-\delta), \tag{5.21}$$

and that $u(t)$ satisfies

$$\lim_{t\to 0}\|u(t)-\phi\|=0, \quad \textit{for some } \phi\in H_0^1(V). \tag{5.22}$$

Then $\frac{\partial z(t)}{\partial t}$ *exists almost everywhere on* $(0,t_1)$ *with* $\frac{\partial z(t)}{\partial t}\in H_0^1(V)$, *and*

$$\begin{aligned}\frac{\partial z(t)}{\partial t} &= \sum_{k=1}^{\infty}\frac{\Phi_k}{\sqrt{\lambda_k}}\sin\left(\sqrt{\lambda_k}t\right)\,(f(\phi),\Phi_k)\\ &+\sum_{k=1}^{\infty}\frac{\Phi_k}{\sqrt{\lambda_k}}\int_0^t\sin\left(\sqrt{\lambda_k}\tau\right)\left(f'(u(t-\tau))\frac{\partial u}{\partial t}(t-\tau),\Phi_k\right)d\tau,\end{aligned} \tag{5.23}$$

for almost all $t\in(0,t_1)$.

Proof. Since $u(t)\in H_0^1(V)$ for all $(0,t_1)$, we see from (1.22) that

$$\|u(t)\|_\infty\le M\,\|u(t)\|\le M\,M_6, \quad \text{for all } t\in(0,t_1), \tag{5.24}$$

where M is as in (1.22) and $M_6>0$. For $\delta>0$ and $t\in(0,t_1-\delta)$,

$$\begin{aligned}&z(t+\delta)-z(t)\\ &=\sum_{k=1}^{\infty}\frac{\Phi_k}{\sqrt{\lambda_k}}\int_t^{t+\delta}\sin\left(\sqrt{\lambda_k}\tau\right)\,(f(u(t+\delta-\tau)),\Phi_k)\,d\tau\\ &+\sum_{k=1}^{\infty}\frac{\Phi_k}{\sqrt{\lambda_k}}\int_0^t\sin\left(\sqrt{\lambda_k}\tau\right)(f(u(t+\delta-\tau))-f(u(t-\tau)),\Phi_k)\,d\tau\\ &\equiv z_1^\delta(t)+z_2^\delta(t).\end{aligned} \tag{5.25}$$

For $t\in(0,t_1-\delta)$, setting $L_1=\sup_{|r|\le M\,M_6}|f'(r)|$,

$$\begin{aligned}\|z_1^\delta(t)\|^2 &=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\left[\int_t^{t+\delta}\sin\left(\sqrt{\lambda_k}\tau\right)\,(f(u(t+\delta-\tau)),\Phi_k)\,d\tau\right]^2\\ &\le\delta\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_t^{t+\delta}(f(u(t+\delta-\tau)),\Phi_k)^2\,d\tau\\ &=\delta\int_t^{t+\delta}\|f(u(t+\delta-\tau))\|_2^2\,d\tau,\end{aligned}$$

where we have used (5.14). Thus, using (5.24) and (5.15),

$$\|z_1^\delta(t)\|\le\frac{L_1M_6}{\sqrt{\lambda_1}}\,\delta, \quad t\in(0,t_1-\delta). \tag{5.26}$$

Similarly,

$$\begin{aligned}\|z_2^\delta(t)\|^2 &=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\Big\{\int_0^t\sin\left(\sqrt{\lambda_k}\tau\right)\,(f(u(t+\delta-\tau))-f(u(t-\tau)),\Phi_k)\,d\tau\Big\}^2\\ &\le s_1^2\left\{\int_0^t\|f(u(t+\delta-\tau))-f(u(t-\tau))\|\,d\tau\right\}^2,\end{aligned}$$

where s_1 satisfies

$$s_1^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty, \tag{5.27}$$

using (3.12) and the fact that $d_s < 2$. By (5.4) and (5.24), we see that

$$\begin{aligned}
&\|f(u(t+\delta-\tau)) - f(u(t-\tau))\| \\
&= \|f'(w_0)(u(t+\delta-\tau) - u(t-\tau))\| \\
&\le \|f'(w_0)\|_\infty \, \|u(t+\delta-\tau) - u(t-\tau)\| \\
&\quad + \|u(t+\delta-\tau) - u(t-\tau)\|_\infty \|f'(w_0)\| \\
&\le (L_1 + M\,\|f'(w_0)\|)\, \|u(t+\delta-\tau) - u(t-\tau)\| \\
&\le M_7\, \|u(t+\delta-\tau) - u(t-\tau)\|,
\end{aligned}$$

for some $w_0 \in H_0^1(V)$ with $\|w_0\|_\infty \le M\,M_6$, and some $M_7 > 0$. Thus, for some $M_8 > 0$,

$$\|z_2^\delta(t)\| \le M_8 \int_0^t \|u(t+\delta-\tau) - u(t-\tau)\|\, d\tau, \quad t \in (0, t_1 - \delta),$$

which combines with (5.26) to give (5.20). If $u(t)$ also satisfies (5.21), then $z(t)$ is Lipschitz continuous in the norm $\|\ \|$ uniformly on $(0, t_1)$, by virtue of (5.20), and so $\partial z(t)/\partial t$ exists in the norm $\|\ \|_2$ almost everywhere on $(0, t_1)$ by Proposition 5.3, and $\frac{\partial z(t)}{\partial t} \in H_0^1(V)$.

It remains to verify (5.23). We have from (5.22) that

$$\lim_{\delta \to 0} \frac{z_1^\delta(t)}{\delta} = \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t)(f(\phi), \Phi_k) \tag{5.28}$$

in the norm $\|\ \|$. Let

$$z_2(t) = \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \int_0^t \sin\left(\sqrt{\lambda_k}\tau\right) \left(f'(u(t-\tau))\, \frac{\partial u}{\partial t}(t-\tau), \Phi_k\right) d\tau.$$

Setting

$$g(t) = \frac{f(u(t+\delta)) - f(u(t))}{\delta} - f'(u(t)) \frac{\partial u(t)}{\partial t},$$

it follows that

$$\begin{aligned}
\left\| \frac{z_2^\delta(t)}{\delta} - z_2(t) \right\|^2 &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left[\int_0^t \sin(\sqrt{\lambda_k}\tau)(g(t-\tau), \Phi_k)\, d\tau \right]^2 \\
&\le \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left[t \int_0^t (g(t-\tau), \Phi_k)^2\, d\tau \right] \\
&= t \int_0^t \|g(t-\tau)\|_2^2\, d\tau. \qquad (5.29)
\end{aligned}$$

Note that for some $\theta_2 \in (0,1)$ and $M_9 > 0$,

$$\begin{aligned}
\|g(t)\|_2 &= \left\| f'(u(t)+\theta_2(u(t+\delta)-u(t)))\frac{u(t+\delta)-u(t)}{\delta} - f'(u(t))\frac{\partial u(t)}{\partial t}\right\|_2 \\
&\le \|f'(u(t)+\theta_2(u(t+\delta)-u(t))) - f'(u(t))\|_2 \left\|\frac{u(t+\delta)-u(t)}{\delta}\right\|_2 \\
&\quad + \|f'(u(t))\|_2 \left\|\frac{u(t+\delta)-u(t)}{\delta} - \frac{\partial u(t)}{\partial t}\right\|_2 \\
&\le M_9 \left\{ \|u(t+\delta)-u(t)\| + \left\|\frac{u(t+\delta)-u(t)}{\delta} - \frac{\partial u(t)}{\partial t}\right\|_2 \right\},
\end{aligned}$$

where we have used (5.15). Hence, it follows from (5.29) that

$$\left\| \frac{z_2^\delta(t)}{\delta} - z_2(t) \right\| \to 0 \quad \text{as } \delta \to 0.$$

Therefore, we obtain (5.23) by virtue of (5.25) and (5.28). ■

We show next that our local solution is smooth in t for all $x \in V$ if the initial data $\phi \in E_\theta(V)$ and $\psi \in E_{\theta-1/2}(V)$ with θ depending on the spectral dimension d_s. In what follows, we denote by $\partial^2 u(t)/\partial^2 t$ the second derivative of u in t in the norm $\| \ \|_2$, that is

$$\lim_{\delta\to 0} \left\| \frac{\frac{\partial u(t+\delta)}{\partial t} - \frac{\partial u(t)}{\partial t}}{\delta} - \partial^2 u(t)/\partial^2 t \right\|_2 = 0.$$

Lemma 5.5 (regularity). *Let $d_s < 2$ and the initial data $\phi \in E_\theta(V)$ and $\psi \in E_{\theta-1/2}(V)$, where $\theta > (6+d_s)/4$. Let $u(t) \in H_0^1(V)$ satisfy (5.6) with $\|u(t)\| \le M_{10}$ on $(0,t_1)$ for some $M_{10} > 0$ and some $t_1 > 0$. Then $u(t)$ has a second derivative $\partial^2 u(t)/\partial^2 t$ in the norm $\| \ \|_2$ for almost all $t \in (0,t_1)$. Moreover,*

$$\Delta u = -f(u(t)) + \frac{\partial^2 u}{\partial^2 t} \tag{5.30}$$

for almost all $t \in (0,t_1)$.

Proof. We first show that $\frac{\partial u(t)}{\partial t} \in H_0^1(V)$ for almost all $t \in (0,t_1)$. Since $\psi = \sum_{k=1}^{\infty} b_k \Phi_k(x) \in E_{\theta-1/2}(V)$, we have that $\sum_{k=1}^{\infty} b_k^2 \lambda_k^{2\theta-2} < \infty$. It follows from (3.13) and (3.12) that

$$\begin{aligned}
\left|\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k} t\right)\ b_k\Phi_k\right| &\le M\sqrt{\lambda_k}|b_k| \\
&\le M\left(\frac{1}{4}|b_k|^2\lambda_k^{2\theta-2} + \lambda_k^{-(2\theta-3)}\right) \\
&\le M\left(\frac{1}{4}|b_k|^2\lambda_k^{2\theta-2} + \beta_4^{2\theta-3}\ k^{-2(2\theta-3)/d_s}\right),
\end{aligned}$$

and so $\sum_{k=1}^{\infty} \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k} t\right)\ b_k\Phi_k$ is uniformly convergent on $(0, t_1) \times V$ since $2(2\theta - 3)/d_s > 1$. Similarly, we have that $\sum_{k=1}^{\infty} \lambda_k \cos\left(\sqrt{\lambda_k} t\right)\ a_k\Phi_k$ is uniformly convergent on $(0, t_1) \times V$ since $\phi = \sum_{k=1}^{\infty} a_k\Phi_k \in E_\theta(V)$ and $\theta > (6 + d_s)/4$. Therefore, we see from (5.7) that $\partial^2 u_0/\partial^2 t$ exists on $(0, t_1) \times V$, and

$$\frac{\partial^2 u_0}{\partial^2 t} = -\sum_{k=1}^{\infty} \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k} t\right)\ b_k\Phi_k\ - \sum_{k=1}^{\infty} \lambda_k \cos\left(\sqrt{\lambda_k} t\right)\ a_k\Phi_k. \tag{5.31}$$

Here the derivative in (5.31) may be understood as the usual partial derivative under the Euclidean metric since they both coincide at this stage.

We claim that $u(t)$ is uniformly Lipschitz continuous in the norm $\|\ \|$ on $(0, t_1)$. To see this, let $\delta > 0$. It follows from (5.6) and Proposition 5.4 that for some $M_{11}, M_{12} > 0$,

$$\begin{aligned} \|u(t+\delta) - u(t)\| &= \|u_0(t+\delta) - u_0(t) + z(t+\delta) - z(t)\| \\ &\le M_{11}\delta + M_{12} \int_0^t \|u(\tau+\delta) - u(\tau)\|\, d\tau \end{aligned} \tag{5.32}$$

for all $t \in (0, t_1 - \delta)$, which yields that, using Gronwall's inequality,

$$\|u(t+\delta) - u(t)\| \le M_{11}\delta\ \exp(M_{12}t) \le M_{13}\delta, \quad t \in (0, t_1 - \delta), \tag{5.33}$$

where $M_{13} = M_{11}\exp(M_{12}t_1)$. This implies that $\partial u(t)/\partial t$ exists in the norm $\|\ \|_2$ for almost all $t \in (0, t_1)$ by Proposition 5.3, and $\left\|\frac{\partial u}{\partial t}(t)\right\| \le M_{13}$ on $(0, t_1)$. Thus $\frac{\partial u}{\partial t}(t) \in H_0^1(V)$ for almost all $t \in (0, t_1)$. Therefore, by (5.6) and Proposition 5.4, for almost all $t \in (0, t_1)$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u_0}{\partial t} + \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k} t\right)\ (f(\phi), \Phi_k) \\ &\quad + \sum_{k=1}^{\infty} \frac{\Phi_k}{\sqrt{\lambda_k}} \int_0^t \sin\left(\sqrt{\lambda_k}\tau\right)\ \left(f'(u(t-\tau))\ \frac{\partial u}{\partial t}(t-\tau), \Phi_k\right)\ d\tau \\ &\equiv v_0(t) + w(t). \end{aligned} \tag{5.34}$$

We show next that $\partial^2 u(t)/\partial^2 t$ exists in the norm $\|\ \|_2$ almost everywhere on $(0, t_1)$. It is easy to see that for $\delta > 0$,

$$\|v_0(t+\delta) - v_0(t)\| \le M_{12}\delta \quad \text{for all } t \in (0, \infty), \tag{5.35}$$

where $M_{12} > 0$. Note that for $g : \mathbb{R} \to \mathbb{R}$,

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_0^t |(g(u(\tau)), \Phi_k)|^2\, d\tau = \int_0^t \|g(u(\tau))\|_2^2\, d\tau. \tag{5.36}$$

Let $v(t) = \partial u/\partial t$ and

$$c_k(t) = \int_0^t \sin(\sqrt{\lambda_k}\tau)(f'(u(t-\tau))v(t-\tau), \Phi_k)\, d\tau, \quad t \in (0, t_1).$$

For $\delta > 0$, we have that for some $M_{13}, M_{14}, M_{15} > 0$, using Hölder's inequality and $(a+b)^2 \leq 2(a^2+b^2), a, b \in \mathbb{R}$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{\lambda_k} |c_k(t+\delta) - c_k(t)|^2 \\
&\leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \Big\{ \int_t^{t+\delta} |(f'(u(t+\delta-\tau))v(t+\delta-\tau), \Phi_k)|\, d\tau \\
&+ \int_0^t |(f'(u(t+\delta-\tau))v(t+\delta-\tau) - f'(u(t-\tau))v(t-\tau), \Phi_k)|\, d\tau \Big\}^2 \\
&\leq M_{13}\delta^2 \\
&+ 4t \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_0^t \Big\{ |([f'(u(t+\delta-\tau)) - f'(u(t-\tau))]v(t+\delta-\tau), \Phi_k)|^2 \\
&+ |(f'(u(t-\tau))[v(t+\delta-\tau) - v(t-\tau)], \Phi_k)|^2 \Big\}\, d\tau \\
&= M_{13}\delta^2 + 4t \int_0^t \Big\{ \|[f'(u(t+\delta-\tau)) - f'(u(t-\tau))]v(t+\delta-\tau)\|_2^2 \\
&\quad + \|f'(u(t-\tau))[v(t+\delta-\tau) - v(t-\tau)]\|_2^2 \Big\}\, d\tau \\
&\leq M_{14}\delta^2 + M_{15} \int_0^t \|v(t+\delta-\tau) - v(t-\tau)\|_2^2\, d\tau
\end{aligned}$$

for all $t \in (0, t_1 - \delta)$, since $u(t), v(t) \in H_0^1(V)$ for almost all $t \in (0, t_1)$. Here we have used (5.36) repeatedly with g replaced by appropriate functions. Therefore,

$$\begin{aligned}
\|w(t+\delta) - w(t)\|^2 &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (c_k(t+\delta) - c_k(t))^2 \\
&\leq M_{14}\delta^2 + M_{15} \int_0^t \|v(t+\delta-\tau) - v(t-\tau)\|_2^2\, d\tau
\end{aligned} \tag{5.37}$$

for all $t \in (0, t_1 - \delta)$. Using the fact that

$$\|v(t+\delta-\tau) - v(t-\tau)\|_2^2 \leq \frac{1}{\lambda_1} \|v(t+\delta-\tau) - v(t-\tau)\|^2,$$

it follows from (5.34), (5.35) and (5.37) that for some $M_{16}, M_{17} > 0$,

$$\|v(t+\delta-\tau) - v(t-\tau)\|^2 \leq M_{16}\delta^2 + M_{17} \int_0^t \|v(t+\delta-\tau) - v(t-\tau)\|^2\, d\tau.$$

Using Gronwall's inequality again, we obtain that for some $M_{18} > 0$,

$$\|v(t+\delta) - v(t)\| \leq M_{18}\delta, \quad t \in (0, t_1 - \delta), \tag{5.38}$$

and so $\partial v/\partial t = \partial^2 u/\partial^2 t$ exists in the norm $\| \ \|_2$ for almost all $t \in (0, t_1)$, again by Proposition 5.3.

It remains to prove (5.30). We see from (5.12) that

$$\frac{\partial z}{\partial t} = \sum_{k=1}^{\infty} \Phi_k \int_0^t \cos\left(\sqrt{\lambda_k}\tau\right) \, (f(u(t-\tau)), \Phi_k) \, d\tau.$$

Since $\partial z/\partial t$ is uniformly Lipschitz on $(0, t_1)$ in the norm $\| \; \|$, we may interchange summation and differentiation, so

$$\frac{\partial^2 z}{\partial^2 t} = \sum_{k=1}^{\infty} \Phi_k \frac{\partial}{\partial t} \int_0^t \cos\left(\sqrt{\lambda_k}\tau\right) \, (f(u(t-\tau)), \Phi_k) \, d\tau,$$

and thus, integrating by parts,

$$\begin{aligned}
\frac{\partial^2 z}{\partial^2 t} &= \sum_{k=1}^{\infty} \cos\left(\sqrt{\lambda_k} t\right) \, (f(\phi), \Phi_k) \Phi_k \\
&\quad + \sum_{k=1}^{\infty} \Phi_k \int_0^t \cos\left(\sqrt{\lambda_k}\tau\right) \, \frac{d}{dt}(f(u(t-\tau)), \Phi_k) \, d\tau \\
&= \sum_{k=1}^{\infty} \cos\left(\sqrt{\lambda_k} t\right) \, (f(\phi), \Phi_k) \Phi_k \\
&\quad - \sum_{k=1}^{\infty} \Phi_k \int_0^t \cos\left(\sqrt{\lambda_k}\tau\right) \, \frac{d}{d\tau}(f(u(t-\tau)), \Phi_k) \, d\tau \\
&= \sum_{k=1}^{\infty} (f(u(t)), \Phi_k) \, \Phi_k \\
&\quad - \sum_{k=1}^{\infty} \Phi_k \int_0^t \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k}\tau\right) \, (f(u(t-\tau)), \Phi_k) \, d\tau \\
&= f(u(t)) - \sum_{k=1}^{\infty} \sqrt{\lambda_k} \Phi_k \int_0^t \sin\left(\sqrt{\lambda_k}(t-\tau)\right) \, (f(u(\tau)), \Phi_k) \, d\tau. \qquad (5.39)
\end{aligned}$$

It follows that the second term on the right-hand side of (5.39) belongs to $H_0^1(V)$ since $\partial^2 z/\partial^2 t, f(u(t)) \in H_0^1(V)$ on $(0, t_1)$. On the other hand, it follows from (3.8) and (5.12) that for $t \in (0, t_1)$,

$$\Delta z(t) = -\sum_{k=1}^{\infty} \sqrt{\lambda_k} \Phi_k \int_0^t \sin\left(\sqrt{\lambda_k}(t-\tau)\right) \, (f(u(\tau)), \Phi_k) \, d\tau,$$

which combines with (5.39) to give

$$\frac{\partial^2 z}{\partial^2 t} = f(u(t)) + \Delta z(t), \quad \text{for almost all } t \in (0, t_1).$$

Clearly,

$$\frac{\partial^2 u_0}{\partial^2 t} = \Delta u_0(t), \quad t > 0.$$

Therefore,

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 z}{\partial^2 t} + \frac{\partial^2 u_0}{\partial^2 t} = f(u(t)) + \Delta u(t),$$

proving (5.30). ■

5.2.3. Global solutions. In this subsection we first obtain a priori estimates on the local solution of (5.6) if f satisfies a growth condition. Then the global existence of a solution of (5.1) and (5.2) immediately follows.

Lemma 5.6 (a priori estimates). *Assume that $f \in C^2(\mathbb{R})$ satisfies*

$$F(r) \leq b_0(1+|r|^2) \quad \text{for all } r \in \mathbb{R} \tag{5.40}$$

for some b_0, where $F(r) = \int_0^r f(s)\,ds$. Let $T > 0$ and $u(t) \in H_0^1(V)$ satisfy (5.6) for $t \in (0,T)$ with the initial data $\phi \in E_\theta(V)$ and $\psi \in E_{\theta-1/2}(V)$, where $\theta > (6+d_s)/4$. Then

$$\sup_{t\in(0,T)} \|u(t)\| \leq M_{19} \tag{5.41}$$

for some M_{19} depending only on T, M_1 and b_0, where M_1 is as in Lemma 5.1.

Proof. Since $u(t) \in H_0^1(V)$ for $t \in (0,T)$, we may write

$$u(t) = \sum_{k=1}^{\infty} a_k(t)\,\varphi_k, \quad t \in (0,T),$$

where $a_k(t) = \int_V u(t,x)\varphi_k(x)\,d\mu(x)$. By Lemma 5.5 we see that $\frac{\partial u(t)}{\partial t}$ and $\frac{\partial^2 u(t)}{\partial^2 t}$ exist almost everywhere on $(0,T)$, and

$$\frac{\partial u(t)}{\partial t} = \sum_{k=1}^{\infty} a_k'(t)\,\varphi_k \in H_0^1(V),$$

$$\frac{\partial^2 u(t)}{\partial^2 t} = \sum_{k=1}^{\infty} a_k''(t)\,\varphi_k \in H_0^1(V),$$

for almost all $t \in (0,T)$. Thus, setting $u_t = \frac{\partial u(t)}{\partial t}$ and $u_{tt} = \frac{\partial^2 u(t)}{\partial^2 t}$,

$$\begin{aligned}\int_V u_t(t,x)\,u_{tt}(t,x)\,d\mu(x) &= \sum_{k=1}^{\infty} a_k'(t)a_k''(t)\\ &= \frac{1}{2}\frac{d}{dt}\sum_{k=1}^{\infty}[a_k'(t)]^2 = \frac{1}{2}\frac{d}{dt}\|u_t(t)\|_2^2\end{aligned}$$

almost everywhere on $(0,T)$, where the interchanging differentiation and summation is valid since the series $\sum_{k=1}^{\infty} a_k'(t)a_k''(t)$ is uniformly convergent on $(0,T)$, using Dirichlet's criterion. Similarly, we have that

$$\begin{aligned}\int_V u_t(t,x)\,\Delta u(t,x)\,d\mu(x) &= -\sum_{k=1}^{\infty} a_k(t)a_k'(t)\lambda_k\\ &= -\frac{1}{2}\frac{d}{dt}\|u(t)\|^2\end{aligned}$$

almost everywhere on $(0,T)$. Therefore, multiplying (5.1) by u_t and integrating over $(0,t)\times V$, we have that

$$\begin{aligned} A(t) &\equiv \frac{1}{2}\left(\|u_t(t)\|_2^2+\|u(t)\|^2\right)-\int_V F(u(t,x))\,d\mu(x) \\ &= \frac{1}{2}\left(\|\psi\|_2^2+\|\phi\|^2\right)-\int_V F(\phi(x))\,d\mu(x)=A(0), \quad t\in(0,T), \end{aligned}$$

and so, using (5.40),

$$\frac{1}{2}\left(\|u_t(t)\|_2^2+\|u(t)\|^2\right)\le A(0)+b_0+b_0\int_V |u(t,x)|^2\,d\mu(x). \tag{5.42}$$

Let $I(t)=\frac{1}{2}\int_V u^2(t,x)\,d\mu(x)$. It follows from (5.42) that

$$\begin{aligned} I'(t) &= \int_V u(t,x)u_t(t,x)\,d\mu(x)\le \|u(t)\|_2\,\|u_t(t)\|_2 \\ &\le \frac{1}{2}\|u(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2 \\ &\le A(0)+b_0+\left(\frac{1}{2}+b_0\right)\|u(t)\|_2^2 \\ &= A(0)+b_0+(1+2b_0)\,I(t), \end{aligned}$$

which means that $I(t)\le M_{20}$ for some $M_{20}>0$ depending only on T, M_1 and b_0, giving

$$\|u(t)\|\le M_{21} \tag{5.43}$$

by virtue of (5.42). ■

Finally, we arrive at the global existence of the solution to (5.1), (5.2).

Theorem 5.7 (global existence). *Let $d_s<2$ and the initial data $\phi\in E_\theta(V)$ and $\psi\in E_{\theta-1/2}(V)$, where $\theta>(6+d_s)/4$. Then (5.1), (5.2) possesses a global weak solution if f satisfies*

$$F(r)\le b_0(1+|r|^2)\quad \textit{for all } r\in\mathbb{R},$$

for some b_0, where $F(r)=\int_0^r f(s)ds$. Such a weak solution $u(t)$ is also a strong solution in the sense that $u(t)$ satisfies (5.1) pointwise for almost all $t\in(0,\infty)$.

Proof. This follows immediately from Lemmas 5.1, 5.5, 5.6. ■

As an example, let $f(r)=-r|r|^p-m_0r$ with $m_0\ge 0$ and $p\ge 2$. Then $f(r)$ satisfies (5.40). Another interesting example satisfying the condition (5.40) is $f(r)=\sin r-m_0\,r, m_0\ge 0$.

5.3. Non-existence of global solutions

In this section we investigate the blow-up of strong solutions to (5.1), (5.2) for a certain class of functions f. We say that u *blows up* if there is $T > 0$ such that

$$\lim_{t\to T-} |u(t,x)| = \infty \quad \text{for some } x \in V. \tag{5.44}$$

We use the method initiated by Levine [**33**] in our fractal situation.

Define the energy $A(t)$ of the strong solution u to (5.1), (5.2) at time t by

$$A(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|u(t)\|^2\right) - \int_V F(u(t,x))\, d\mu(x), \quad t > 0, \tag{5.45}$$

where $F(r) = \int_0^r f(s)\, ds$. The basic fact about nonlinear wave equations is the conservation of energy, that is

$$A(t) = A(0), \quad \text{for all } t > 0. \tag{5.46}$$

This is easily seen by multiplying (5.1) by u_t and then integrating on $(0,t) \times V$.

Theorem 5.8 (blow-up). *Assume that the initial data ϕ and ψ satisfies $A(0) <$ 0, and that f satisfies*

$$rf(r) \geq (2+\varepsilon)F(r) \quad \text{for all } r \in \mathbb{R} \tag{5.47}$$

for some $\varepsilon > 0$. Then there exists $T > 0$ such that a strong solution u to (5.1), (5.2) on $(0,T) \times V$ satisfies (5.44).

Proof. The proof given here is motivated by [**33**]. Let

$$G(t) = \frac{1}{2}\int_V u(t,x)^2\, d\mu(x) + 1 + s_0(t+\gamma)^2, \quad t \geq 0,$$

where $s_0 > 0$ and γ are constants to be determined below. We calculate that for $t > 0$,

$$\begin{aligned}
&G(t)\, G''(t) - (s_2+1)(G'(t))^2 \\
&= 2\left(\frac{1}{2}\|u(t)\|_2^2 + 1 + s_0(t+\gamma)^2\right)\left(\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\int_V u(t,x)u_{tt}(t,x)\, d\mu(x) + s_0\right) \\
&\quad - 4(s_2+1)\left(\frac{1}{2}\int_V u(t,x)u_t(t,x)\, d\mu(x) + s_0(t+\gamma)\right)^2 \\
&= 4(s_2+1)\Bigg\{\left(\frac{1}{2}\|u(t)\|_2^2 + s_0(t+\gamma)^2\right)\left(\frac{1}{2}\|u_t(t)\|_2^2 + s_0\right) \\
&\quad - \left(\frac{1}{2}\int_V u(t,x)u_t(t,x)\, d\mu(x) + s_0(t+\gamma)\right)^2\Bigg\} + 4(s_2+1)\left(\frac{1}{2}\|u_t(t)\|_2^2 + s_0\right) \\
&\quad + 2G(t)\left(\frac{1}{2}\int_V u(t,x)u_{tt}(t,x)\, d\mu(x) - (2s_2+1)\left(\frac{1}{2}\|u_t(t)\|_2^2 + s_0\right)\right).
\end{aligned} \tag{5.48}$$

Note that the first term on the right-hand side of (5.48) is non-negative since, using Hölder's and Cauchy's inequalities,

$$\begin{aligned}
&\left(\frac{1}{2}\int_V u(t,x)u_t(t,x)\,d\mu(x) + s_0(t+\gamma)\right)^2 \\
&\le \left(\frac{1}{2}\|u(t)\|_2\,\|u_t(t)\|_2 + s_0(t+\gamma)\right)^2 \\
&\le \left(\frac{1}{2}\|u(t)\|_2^2 + s_0(t+\gamma)^2\right)\left(\frac{1}{2}\|u_t(t)\|_2^2 + s_0\right).
\end{aligned}$$

Multiplying (5.1) by u and integrating over V, we have that for $t > 0$,

$$\int_V u(t,x)\,u_{tt}(t,x)\,d\mu(x) = -\|u(t)\|^2 + \int_V u(t,x)f(u(t,x))\,d\mu(x),$$

and so, using (5.45) and (5.46),

$$\begin{aligned}
&\frac{1}{2}\int_V u(t,x)\,u_{tt}(t,x)\,d\mu(x) - (2s_2+1)\left(\frac{1}{2}\|u_t(t)\|_2^2 + s_0\right) \\
&= -\frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\int_V u(t,x)f(u(t,x))\,d\mu(x) - \left(s_2 + \frac{1}{2}\right)\|u_t(t)\|_2^2 - (2s_2+1)s_0 \\
&= -(2s_2+1)(A(0)+s_0) + s_2\|u(t)\|^2 \\
&\quad + \frac{1}{2}\int_V \left(u(t,x)f(u(t,x)) - 2(2s_2+1)F(u(t,x))\right)\,d\mu(x) \ge 0
\end{aligned}$$

if we choose $s_0 = -A(0) > 0$ and $s_2 = \varepsilon/4$ by virtue of (5.47). Therefore, we have from (5.48) that for $t > 0$,

$$G(t)\,G''(t) - (s_2+1)(G'(t))^2 \ge 0,$$

and so, setting $Q(t) = G(t)^{-s_2}$,

$$Q''(t) = -s_2 G(t)^{-(s_2+2)}\left(G(t)\,G''(t) - (s_2+1)(G'(t))^2\right) \le 0.$$

Hence,

$$Q(t) \le Q(0) + Q'(0)t, \quad t > 0,$$

that is

$$G(t)^{-s_2} \le G(0)^{-s_2}\left(1 - s_2 G'(0)t/G(0)\right).$$

Note that $G'(0) = \int_V \phi(x)\psi(x)\,d\mu(x) + 2s_0\gamma > 0$ if we take

$$\gamma > -\frac{1}{2s_0}\int_V \phi(x)\psi(x)\,d\mu(x).$$

Thus there exists $T > 0$ satisfying

$$T \le \frac{4G(0)}{\varepsilon G'(0)},$$

such that $\lim_{t\to T-} G(t) = \infty$, proving (5.44). ■

An example for which (5.47) holds is $f(r) = r|r|^{p-1} - m_0\,r, p > 2$ and $m_0 \ge 0$.

CHAPTER 6

Nonlinear diffusion equations on p.c.f. self-similar fractals

Let V be a p.c.f. self-similar fractal that has a harmonic structure and satisfies the separation condition (1.17), with the boundary V_0. Let μ be a Borel measure on V normalized so that $\mu(V) = 1$. In particular the Sobolev-type inequality (1.13) holds on V. In this chapter we investigate the nonlinear diffusion equations

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad t > 0,\ x \in V \backslash V_0, \tag{6.1}$$

with given initial data and zero boundary conditions

$$\begin{aligned} u|_{t=0} &= u_0(x), \quad x \in V, \\ u|_{V_0} &= 0, \quad t \geq 0, \end{aligned} \tag{6.2}$$

where Δ is the weak Laplacian defined by (1.23). The function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz continuous. We suppose that the initial data u_0 lies in $C_0(V)$, the space of all continuous functions on V vanishing on V_0. We show that (6.1) and (6.2) possess a 'global' solution for suitable f and small initial data by employing the iteration scheme and a maximum principle, that is (6.1)-(6.2) has a solution for all $t > 0$. The main result in this chapter appeared in [**21**].

6.1. The maximum principle

In this section we establish a maximum principle on V.

Define the family of mappings $\{P_t : t > 0\}$ on $L^2(V)$ by

$$P_t u(x) = \int_V K(t,x,y) u(y)\, d\mu(y), \quad t > 0 \text{ and } x \in V \tag{6.3}$$

for $u \in L^2(V)$, see (3.10), where K is as in (3.11):

$$K(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x) \varphi_k(y), \quad t > 0 \text{ and } x, y \in V.$$

Clearly each P_t is linear and symmetric, and satisfies the semigroup property

$$P_t P_s = P_{t+s}, \quad t, s > 0.$$

Moreover, we have

Proposition 6.1. *Each $P_t (t > 0)$ is a contraction on $L^2(V)$, that is*

$$\|P_t u\|_2 \leq \|u\|_2, \quad u \in L^2(V). \tag{6.4}$$

Moreover, for $u \in C_0(V)$,

$$\lim_{t \downarrow 0} \|P_t u - u\|_\infty \equiv \lim_{t \downarrow 0} \sup_{x \in V} |P_t u(x) - u(x)| = 0. \tag{6.5}$$

Proof. Let $\{\varphi_k\}_{k \geq 1}$ be the complete orthonormal basis of $L^2(V)$, see Chapter 3. Let $u \in L^2(V)$. By Parseval's relation,

$$\|u\|_2^2 = \sum_{k=1}^{\infty} a_k^2,$$

where $a_k = \int_V f(x)\varphi_k(x)\, d\mu(x)$. It follows from (6.3) that for $t > 0$,

$$\begin{aligned}\|P_t u\|_2^2 &= \sum_{k=1}^{\infty} a_k^2 \exp(-2\lambda_k t) \\ &\leq \sum_{k=1}^{\infty} a_k^2 = \|u\|_2^2,\end{aligned}$$

giving (6.4).

For $u = \sum_{k=1}^{\infty} a_k \varphi_k \in H_0^1(V)$,

$$\begin{aligned}\|P_t u - u\|^2 &= W\left(P_t u - u, P_t u - u\right) \\ &= \sum_{k=1}^{\infty} a_k^2 (\exp(-\lambda_k t) - 1)^2 \lambda_k \to 0 \quad \text{as } t \downarrow 0,\end{aligned} \tag{6.6}$$

since the series

$$\sum_{k=1}^{\infty} a_k^2 (\exp(-\lambda_k t) - 1)^2 \lambda_k \leq 4 \sum_{k=1}^{\infty} a_k^2 \lambda_k = 4W(u,u) < \infty$$

is uniformly convergent in $t \geq 0$, giving (6.5) for $u \in H_0^1(V)$, by virtue of (1.22). For $u \in C_0(V)$, there is a sequence of $\{u_j\} \in H_0^1(V)$ such that

$$\|u_j - u\|_\infty \to 0, \quad j \to \infty.$$

Thus

$$\begin{aligned}\lim_{t \to 0} \|P_t u - u\|_\infty &\leq \lim_{t \to 0} \Big(\|P_t u - P_t u_j\|_\infty \\ &\qquad + \|P_t u_j - u_j\|_\infty + \|u_j - u\|_\infty \Big) \\ &\leq \lim_{t \to 0} (2\|u_j - u\|_\infty + \|P_t u_j - u_j\|_\infty) \\ &= 2\|u_j - u\|_\infty,\end{aligned} \tag{6.7}$$

since

$$\|P_t u - P_t u_j\|_\infty \leq \|u_j - u\|_\infty.$$

Note that the left-hand side in (6.7) is independent of j. We let $j \to \infty$ to get (6.5) for $u \in C_0(V)$. ∎

Let Δ be the weak Laplacian given by (1.23). It is easy to see that

$$\Delta u = \lim_{h \downarrow 0} h^{-1}(P_h u - u), \tag{6.8}$$

where the limit is taken in the L^2-norm, recall (3.9). Thus (6.8) gives an alternative definition of the weak Laplacian on V, provided that the heat kernel K is known. Clearly

$$\Delta \varphi_k(x) = -\lambda_k \varphi_k(x) \quad \text{pointwise in } V \backslash V_0, \; k \geq 1. \tag{6.9}$$

From (6.3) and (3.11), it is easily seen that P_t is self-adjoint, that is for $u, v \in L^2(V)$

$$\int_V P_t u(x) v(x) \, d\mu(x) = \int_V P_t v(x) u(x) \, d\mu(x)$$

and so

$$\begin{aligned} \int_V \Delta u(x) v(x) \, d\mu(x) &= \lim_{h \downarrow 0} h^{-1} \int_V (P_h u - u) v(x) \, d\mu(x) \\ &= \lim_{h \downarrow 0} h^{-1} \int_V u(x)(P_h v - v) \, d\mu(x) \\ &= \int_V \Delta v(x) u(x) \, d\mu(x). \end{aligned}$$

Therefore, for $u, v \in L^2(V)$ and $\Delta u, \Delta v \in L^2(V)$ we get the Gauss-Green formula

$$\int_V \Delta u(x) v(x) \, d\mu(x) = \int_V \Delta v(x) u(x) \, d\mu(x). \tag{6.10}$$

Proposition 6.2. *Let K be as in (3.11). Then for all $x \in V$ and $t_0 > 0$, $y_0 \in V$, $\frac{\partial K}{\partial t}(t_0, x, y_0)$ exists and*

$$\frac{\partial K}{\partial t}(t_0, x, y_0) = \Delta K(t_0, x, y_0). \tag{6.11}$$

Proof. Let $t_0 > 0$. Note that the series $\sum\limits_{k=1}^{\infty} \lambda_k \exp(-\lambda_k t_0) \varphi_k(x) \varphi_k(y_0)$ is uniformly convergent for all $x, y_0 \in V$. Thus $\frac{\partial K}{\partial t}(t_0, x, y_0)$ exists and

$$\frac{\partial K}{\partial t}(t_0, x, y_0) = -\sum_{k=1}^{\infty} \lambda_k \exp(-\lambda_k t_0) \varphi_k(x) \varphi_k(y_0) \tag{6.12}$$

for all $x \in V$ and $t_0 > 0$, $y_0 \in V$. On the other hand, we see that for fixed t_0, y_0, using (3.11) and (6.9),

$$\Delta K(t_0, x, y_0) = -\sum_{k=1}^{\infty} \lambda_k \exp(-\lambda_k t_0) \varphi_k(x) \varphi_k(y_0)$$

giving (6.11). ∎

We state the maximum principle on V.

Proposition 6.3. *Let $T > 0$. Assume that $v(t,.) \in \mathcal{D}(\Delta)$ on $(0,T]$, and $v(t,x)$ is continuous on $[0,T] \times V$ and satisfies*

$$\begin{aligned} &\Delta v - \delta_1 v - \frac{\partial v}{\partial t} \le 0, \quad t \in (0,T], \quad x \in V \backslash V_0, \\ &v|_{t=0} = v_0(x) \ge 0, \quad x \in V, \\ &v|_{V_0} = 0, \quad t \ge 0, \end{aligned} \tag{6.13}$$

where $\delta_1 > 0$. Then

$$v(t,x) \ge 0 \quad \text{for } (t,x) \in (0,T] \times V, \tag{6.14}$$

provided that $\Delta v(t,x)$ is continuous on $(0,T] \times V$.

Proof. Suppose $(t_0, x_0) \in (0,T] \times V$ is such that $v(t_0,x_0) < 0$. Since $v(t,x)$ is continuous on $[0,T] \times V$ and $v_0(x) \ge 0$, there must exist $(t_1,x_1) \in (0,T] \times V$ such that v reaches its negative minimum at (t_1,x_1). Note that $\frac{\partial v}{\partial t}(t_1,x_1) \le 0$. We claim that

$$\Delta v(t_1,x_1) \ge 0. \tag{6.15}$$

To see this, note that for $h > 0$,

$$\begin{aligned} \frac{\partial}{\partial h} P_h v(t_1,x_1) &= \frac{\partial}{\partial h} \int_V K(h,x_1,y) v(t_1,y)\, d\mu(y) \\ &= \int_V \frac{\partial}{\partial h} K(h,x_1,y) v(t_1,y)\, d\mu(y) \\ &= \int_V \Delta K(h,x_1,y) v(t_1,y)\, d\mu(y) \\ &= \int_V K(h,x_1,y) \Delta v(t_1,y)\, d\mu(y). \end{aligned}$$

Integrating on $(0,h)$ and then using (6.5), it follows that for $h > 0$,

$$P_h v(t_1,x_1) - v(t_1,x_1) = \int_0^h d\tau \int_V K(\tau,x_1,y) \Delta v(t_1,y)\, d\mu(y). \tag{6.16}$$

Observe that for all $h > 0$, using (3.16) and (3.17),

$$P_h v(t_1,x_1) \ge v(t_1,x_1) \int_V K(h,x_1,y)\, d\mu(y) \ge v(t_1,x_1),$$

since (t_1,x_1) is the minimum point of v. Therefore, by (6.16),

$$\frac{1}{h} \int_0^h d\tau \int_V K(\tau,x_1,y) \Delta v(t_1,y)\, d\mu(y) \ge 0,$$

for all small $h > 0$. Letting $h \to 0$ and then using the continuity of Δv, we get (6.15). Therefore,

$$0 \le \Delta v(t_1,x_1) - \frac{\partial v}{\partial t}(t_1,x_1) \le \delta_1 v(t_1,x_1) < 0.$$

But this is a contradiction, proving (6.14). ∎

Kigami [**23**] obtained a maximum principle with $\delta_1 = 0$ in (6.13).

Corollary 6.4. *Let $T > 0$. Let $w(t,.) \in \mathcal{D}(\Delta)$ be continuous on $[0,T] \times V$ and satisfy*

$$\begin{aligned} \Delta w - \delta_2 w - \frac{\partial w}{\partial t} &\geq 0, \quad t > 0, \quad x \in V \backslash V_0, \\ w|_{t=0} = w_0(x) &\leq 0, \quad x \in V, \\ w|_{V_0} &= 0, \quad t \geq 0, \end{aligned} \tag{6.17}$$

where $\delta_2 > 0$. Then

$$w(t,x) \leq 0 \quad \text{for } (t,x) \in (0,T] \times V \tag{6.18}$$

provided that Δw is continuous on $(0,T] \times V$.

Proof. Let $v(t,x) = -w(t,x)$ and (6.17) follows immediately from Proposition 6.3. ■

6.2. Existence of solutions

We establish the existence of solutions to (6.1), (6.2) for suitable f and small initial data by using the iteration scheme and the maximum principle. To do this, we first investigate the linear problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u, \quad t > 0, x \in V \backslash V_0, \\ u|_{t=0} &= \phi(x), \quad x \in V, \\ u|_{V_0} &= 0, \quad t \geq 0. \end{aligned} \tag{6.19}$$

Clearly (6.19) has a unique solution

$$u(t,x) = \int_V K(t,x,y)\phi(y)\, d\mu(y) \tag{6.20}$$

for $\phi \in L^2(V)$.

Proposition 6.5. *Let u be the solution of the linear problem (6.19). If the initial data $\phi \in C_0(V)$, then $u(t,x)$ is continuous on $[0,\infty) \times V$.*

Proof. Since u is the solution of (6.19), we see that

$$\begin{aligned} u(t,x) &= \int_V K(t,x,y)\phi(y)\, d\mu(y) \\ &= \sum_{k=1}^{\infty} a_k \exp(-\lambda_k t)\varphi_k(x), \quad t > 0 \text{ and } x \in V, \end{aligned} \tag{6.21}$$

where $a_k = \int_V \varphi_k(y)\phi(y)\ d\mu(y)$. It is easily seen that $u(t,x)$ is continuous on $(0,\infty) \times V$ since

$$\sum_{k=1}^{\infty} a_k \exp(-\lambda_k t)\varphi_k(x)$$

is uniformly convergent for all $x \in V$ and $t \geq \eta > 0$. It remains to prove that $u(t,x)$ is continuous at $\{0\} \times V$. But this follows from (6.5). ■

Corollary 6.6. *Let v be the solution of the linear diffusion equation*

$$\begin{aligned}\frac{\partial v}{\partial t} + \delta_3 v &= \Delta v + h(t,x) \quad t > 0, \quad x \in V \backslash V_0,\\ v|_{t=0} &= v_0(x), \quad x \in V,\\ v|_{V_0} &= 0, \quad t \geq 0,\end{aligned}$$

where δ_3 is a constant and $h(t,x)$ is continuous on $[0,\infty) \times V$. Then the solution v is continuous on $[0,\infty) \times V$ if the initial data $v_0 \in C_0(V)$.

Proof. Let $w(t,x) = v(t,x)\exp(\delta_3 t)$. Then w satisfies

$$\begin{aligned}&\frac{\partial w}{\partial t} = \Delta w + \exp(\delta_3 t)h(t,x), \quad t > 0, x \in V \backslash V_0,\\ &w|_{t=0} = v_0(x), \quad x \in V,\\ &w|_{V_0} = 0, \quad t \geq 0.\end{aligned}$$

Therefore,

$$w(t,x) = u(t,x) + \int_0^t d\tau \int_V K(t-\tau,x,y)\exp(\delta_3\tau)h(\tau,y)\ d\mu(y), \tag{6.22}$$

where $u(t,x)$ is the solution of (6.19) with the initial data v_0. From Proposition 6.5, $u(t,x)$ is continuous on $[0,\infty) \times V$ since $v_0 \in C_0(V)$. The second term on the right-hand side of (6.22) is also continuous on $[0,\infty) \times V$ since $h(t,x)$ is continuous on $[0,\infty) \times V$. ■

We require the concepts of upper and lower solutions. Let $T > 0$ and $\Gamma_T = (0,T] \times V$. A function $u_1 : \Gamma_T \to \mathbb{R}$ is an *upper solution* of (6.1), (6.2) on Γ_T if

$$\begin{aligned}\Delta u_1 + f(u_1) &- \frac{\partial u_1}{\partial t} \leq 0 \quad \text{in } \Gamma_T,\\ u_1|_{t=0} &\geq u_0(x), \quad x \in V,\\ u_1|_{V_0} &= 0, \quad t \geq 0.\end{aligned} \tag{6.23}$$

A function $v_1 : \Gamma_T \to \mathbb{R}$ is a *lower solution* of (6.1), (6.2) on Γ_T if

$$\begin{aligned}\Delta v_1 + f(v_1) &- \frac{\partial v_1}{\partial t} \geq 0 \quad \text{in } \Gamma_T,\\ v_1|_{t=0} &\leq u_0(x), \quad x \in V,\\ v_1|_{V_0} &= 0, \quad t \geq 0.\end{aligned} \tag{6.24}$$

As before, Δ is the generator of the semigroup $\{P_t : t > 0\}$ associated with the heat kernel K.

Given upper and lower solutions u_1 and v_1 in Γ_T with $v_1 \leq u_1$, we choose $\Theta > 0$ so large that $\Theta > L_f$, where L_f is the Lipschitz constant of f, that is

$$|f(w_2) - f(w_1)| \leq L_f |w_2 - w_1|$$

for $w_1, w_2 : \Gamma_T \to \mathbb{R}$ such that $\min_{\Gamma_T} v_1 \leq w_1, w_2 \leq \max_{\Gamma_T} u_1$. Let $z_1 : \Gamma_T \to \mathbb{R}$ be continuous and $v_1 \leq z_1 \leq u_1$. We define z_2 by

$$\begin{aligned} \Delta z_2 - \Theta z_2 &- \frac{\partial z_2}{\partial t} = -(f(z_1) + \Theta z_1) \quad \text{in } \Gamma_T, \\ z_2|_{t=0} &= u_0(x), \quad x \in V, \\ z_2|_{V_0} &= 0, \quad t \geq 0. \end{aligned} \tag{6.25}$$

From Corollary 6.6, the solution z_2 of (6.25) is continuous on $[0,T]\times V$ if $u_0 \in C_0(V)$. Using Propositon 6.3 and (6.23), (6.25), we see that $z_2 \leq u_1$ in Γ_T. Similarly, we have that $v_1 \leq z_2$ by using Corollary 6.4 and (6.24), (6.25).

Let $\mathcal{F}$ be the mapping given by $z_2 = \mathcal{F} z_1$, where z_2 is the solution of (6.25) corresponding to z_1. Let $\mathcal{B} = \{z | z : \Gamma_T \to \mathbb{R} \text{ and } v_1 \leq z \leq u_1\}$. Then $\mathcal{F}$ is a mapping from $\mathcal{B}$ to $\mathcal{B}$.

Proposition 6.7. *$\mathcal{F}$ is a monotone mapping in the sense of Collatz, that is*

$$\mathcal{F}u \leq \mathcal{F}v \quad \text{if } u \leq v \tag{6.26}$$

for $\min v_1 \leq u, v \leq \max u_1$.

Proof. Let $u \leq v$ for $\min v_1 \leq u, v \leq \max u_1$. Then

$$\begin{aligned} \Delta \mathcal{F}u - \Theta \mathcal{F}u &- \frac{\partial \mathcal{F}u}{\partial t} = -(f(u) + \Theta u) \quad \text{in } \Gamma_T, \\ \mathcal{F}u|_{t=0} &= u_0(x), \quad x \in V, \\ \mathcal{F}u|_{V_0} &= 0, \quad t \geq 0, \end{aligned}$$

and

$$\begin{aligned} \Delta \mathcal{F}v - \Theta \mathcal{F}v &- \frac{\partial \mathcal{F}v}{\partial t} = -(f(v) + \Theta v) \quad \text{in } \Gamma_T, \\ \mathcal{F}v|_{t=0} &= u_0(x), \quad x \in V, \\ \mathcal{F}v|_{V_0} &= 0, \quad t \geq 0. \end{aligned}$$

Therefore, setting $w = \mathcal{F}v - \mathcal{F}u$,

$$\begin{aligned} \Delta w - \Theta w &- \frac{\partial w}{\partial t} = -(f(v) - f(u) + \Theta(v-u)) \quad \text{in } \Gamma_T, \\ w|_{t=0} &= 0, \quad x \in V, \\ w|_{V_0} &= 0, \quad t \geq 0. \end{aligned}$$

Since $u \leq v$ and $\Theta > L_f$, we see that $f(v) - f(u) + \Theta(v-u) \geq 0$. Thus by Proposition 6.3 we have that $w \geq 0$, giving (6.26). ■

We now obtain a solution to (6.1), (6.2) by an iteration procedure.

Lemma 6.8. *If $u_0 \in C_0(V)$ and there are upper and lower solutions u_1 and v_1 of (6.1), (6.2) satisfying (6.23) and (6.24) respectively, then there is a function u satisfying*

$$\begin{aligned} u(t,x) &= \int_V K(t,x,y)u_0(y)\,d\mu(y) \\ &\quad + \int_0^t d\tau \int_V K(t-\tau,x,y)f(u(\tau,y))\,d\mu(y) \end{aligned} \tag{6.27}$$

with the property that $v_1 \le u \le u_1$ in Γ_T, where $T > 0$.

Proof. Inductively, we define $u_m : \Gamma_T \to \mathbb{R}$ by $u_{m+1} = \mathcal{F}u_m$ $(m \ge 1)$, where u_1 is the upper solution of (6.1), (6.2). Since $\mathcal{F}$ is monotone and $u_2 \le u_1$, we see that $u_{m+1} = \mathcal{F}u_m \le \mathcal{F}u_{m-1} = u_m$ for all $m \ge 2$, that is the sequence $\{u_m\}$ is decreasing in m for all $(t,x) \in \Gamma_T$. On the other hand, we define $v_{m+1} = \mathcal{F}v_m$ $(m \ge 1)$ where v_1 is a lower solution; it follows by Corollary 6.4 that $v_2 \ge v_1$. Thus the sequence $\{v_m\}$ is increasing in m for all $(t,x) \in \Gamma_T$. Moreover, $v_m \le u_m$ for all $m \ge 1$ since $v_1 \le u_1$, and $v_m = \mathcal{F}v_{m-1} \le \mathcal{F}u_{m-1} = u_m$ if $v_{m-1} \le u_{m-1}$. Thus

$$v_1 \le u_m \le u_1, \quad \text{in } \Gamma_T \quad \text{for } m \ge 1. \tag{6.28}$$

Therefore, there exists $u : \Gamma_T \to \mathbb{R}$ with the property $v_1 \le u \le u_1$ such that

$$\lim_{m\to\infty} u_m(t,x) = u(t,x) \quad \text{pointwise in } \Gamma_T. \tag{6.29}$$

We have

$$\begin{aligned} u_{m+1}(t,x) = \mathcal{F}u_m(t,x) &= \int_V K(t,x,y)u_0(y)\,d\mu(y) \\ + \int_0^t d\tau \int_V K(t-\tau,x,y)&\Big[f(u_m(\tau,y)) + \Theta(u_m(\tau,y) - u_{m+1}(\tau,y))\Big]\,d\mu(y), \end{aligned}$$

giving (6.27) by letting $m \to \infty$ and using the dominated convergence theorem.■

Proposition 6.9. *Let u be bounded and satisfy (6.27). Suppose that $f \in C_1(\mathbb{R})$ and $u_0 \in C_0(V)$ is such that*

$$\frac{\partial}{\partial t}P_t u_0 \quad \text{exists and is bounded for all } t > 0 \quad \text{and all } x \in V, \tag{6.30}$$

where $P_t u_0 = \int_V K(t,x,y)u_0(y)\,d\mu(y)$. Then $u(t,x)$ satisfies (6.1) pointwise, where Δ is the generator of the semigroup $\{P_t : t > 0\}$ associated with the heat kernel K.

Proof. Set $u_0(t,x) = P_t u_0(x)$. Since u satisfies (6.27), we have that for $\delta > 0$,

$$\begin{aligned}
&u(t+\delta,x) - u(t,x)\\
&= u_0(t+\delta,x) - u_0(t,x) + \int_0^{t+\delta} d\tau \int_V K(\tau,x,y) f(u(t+\delta-\tau,y))\, d\mu(y)\\
&\quad - \int_0^t d\tau \int_V K(\tau,x,y) f(u(t-\tau,y))\, d\mu(y)\\
&= u_0(t+\delta,x) - u_0(t,x)\\
&\quad + \int_0^t d\tau \int_V K(\tau,x,y)\Big[f(u(t+\delta-\tau,y)) - f(u(t-\tau,y))\Big]\, d\mu(y)\\
&\quad + \int_t^{t+\delta} d\tau \int_V K(\tau,x,y) f(u(t+\delta-\tau,y))\, d\mu(y).
\end{aligned}$$

Letting

$$g(t) = \sup_{x\in V} |u(t+\delta,x) - u(t,x)|, \quad t > 0,$$

we see that, using (3.17) and (6.30),

$$g(t) \le \delta_4 \left(\delta + \int_0^t g(t-\tau)\, d\tau\right), \quad t > 0$$

since f is Lipschitz and u is bounded, where δ_4 is a constant. Applying Gronwall's inequality, it follows that

$$g(t) \le \delta_4 \delta \exp(\delta_4 t), \quad t > 0, \tag{6.31}$$

which implies $u(t,x)$ is uniformly Lipschitz on $t \in (0,T]$ for all $x \in V$ and all $T > 0$, and so $\frac{\partial u}{\partial t}$ exists for almost every $t > 0$ and all $x \in V$. Thus the second term on the right-hand side of (6.27) is differentiable with respect to $t > 0$ and its derivative equals

$$\int_V K(t,x,y) f(u_0(y))\, d\mu(y)$$
$$\qquad + \int_0^t d\tau \int_V K(\tau,x,y) \frac{\partial f(u)}{\partial u}(t-\tau,y) \frac{\partial u}{\partial t}(t-\tau,y)\, d\mu(y)$$

for all $x \in V$ and $t > 0$. It is not hard to verify that Δu exists for all $t > 0$ and all $x \in V$ since $\frac{\partial u}{\partial t}$ exists for all $t > 0$ and all $x \in V$, and

$$\Delta u(t,x) = \frac{\partial u}{\partial t}(t,x) - f(u(t,x))$$

for all $t > 0$ and all $x \in V$. ■

Note that if

$$u_0(x) = \int_V w_0(y) K(\delta,x,y_0)\, d\mu(y),$$

where $\delta > 0$ and $w_0 \in L^1(V)$, then u_0 satisfies (6.30). Another example when (6.30) holds is that $u_0(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \in L^2(V)$ with $\sum_{k=1}^{\infty} |a_k| \lambda_k^{3/2} < \infty$.

Theorem 6.10. *Suppose that* $|f(r)| \leq \lambda_1 |r|$ *for* $|r| \leq b_0$ *for some* $b_0 > 0$, *and that* $u_0 \in C_0(V)$ *satisfies* $|u_0(x)| \leq \delta_5 \varphi_1(x)$ *in* V, *where* λ_1 *is the smallest eigenvalue of (3.2) with eigenfunction* φ_1 *and* δ_5 *is so small that* $\max \varphi_1 \leq b_0/\delta_5$. *Then there exists* u *satisfying (6.27). Moreover, if* $f \in C_1$ *and the initial data* u_0 *satisfies (6.30), then* $u(t,x)$ *satisfies (6.1) pointwise for all* $t > 0$ *and all* $x \in V \backslash V_0$.

Proof. The proof here is based on [**16**]. Note that the eigenfunction φ_1 in (3.2) can be taken to be non-negative on V, see Corollary 2.2. Let $u_1(t,x) = \delta_5 \varphi_1(x)$. Then

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \lambda_1 u_1 &= \Delta u_1, \quad t > 0, \quad x \in V \backslash V_0, \\ u_1|_{t=0} &= \delta_5 \varphi_1(x), \quad x \in V, \\ u_1|_{V_0} &= 0, \quad t \geq 0. \end{aligned}$$

It is not hard to verify that u_1 is an upper solution. Similarly, $v_1(t,x) = -\delta_5 \varphi_1(x)$ is a lower solution. The result follows immediately from Lemma 6.8 and Proposition 6.9. ∎

For a specific example, let $f(r) = r|r|^{p-1} (p > 1)$. Then (6.1), (6.2) has a global solution if the initial data is sufficiently small.

CHAPTER 7

Nonlinear diffusion equations on unbounded fractal domains

In this chapter the domain is no longer restricted to be a p.c.f. self-similar fractal, and so we use G instead of V to denote the domain. Also, we take a different definition for a heat kernel from that discussed in the earlier chapters, which will be denoted by the small letter k.

We investigate the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \qquad p > 1,$$

on certain unbounded fractal domains, where Δ is the infinitesimal generator of the semigroup associated with a corresponding heat kernel. We show that there are non-negative global solutions for non-negative initial data if $p > 1 + \frac{2}{d_s}$, while solutions blow up if $p \leq 1 + \frac{2}{d_s}$, where d_s is the *spectral dimension* of the domain, see Section 7.1. We investigate smoothness and Hölder continuity of solutions when they exist. The results of this chapter appeared in [**13**].

7.1. Preliminaries

Let G be an unbounded domain in $\mathbb{R}^n (n \geq 2)$ which will generally be a fractal. We consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad t > 0, x \in G, p > 1, \tag{7.1}$$

with non-negative initial values

$$u|_{t=0} = \phi(x), \quad x \in G. \tag{7.2}$$

Our approach in this chapter is through a family of integral operators defined in terms of a heat kernel. We first consider 'weak solutions' to (7.1)-(7.2), by which, in this chapter, we mean solutions of a corresponding integral equation involving a heat kernel k on G. The Laplacian may then be defined as the infinitesimal generator of the associated semigroup, to enable us to investigate 'strong solutions'. However, the existence of a heat kernel with suitable properties on a fractal is a non-trivial question, and in general such heat kernels cannot be expressed explicitly.

We consider the existence and non-existence of non-negative global solutions to (7.1)-(7.2), and the regularity properties of non-negative bounded global solutions when they exist. The term "global" implies a solution $u : G \times (0, \infty) \to \mathbb{R}$, that is existing for all $t > 0$.

Let μ be a locally finite Borel measure on G. We are particularly interested in the case where G is an unbounded fractal (such as the unbounded Sierpínski gasket) when μ might typically be d_f-dimensional Hausdorff measure if G has Hausdorff dimension d_f, see [**9**].

We term a continuous $k : (0, \infty) \times G \times G \to \mathbb{R}$ a *heat kernel* if it satisfies

(K_1): (Positivity) $k(t, x, y) > 0$ for all $t > 0$ and all $(x, y) \in G \times G$;

(K_2): (Symmetry) $k(t, x, y) = k(t, y, x)$ for all $t > 0$ and all $(x, y) \in G \times G$;

(K_3): (normalization) $\int_G k(t, x, y) d\mu(y) = 1$ for all $t > 0$ and all $x \in G$;

(K_4): (Semigroup property) $k(s + t, x, y) = \int_G k(t, x, z) k(s, z, y) d\mu(z)$ for all $t, s > 0$ and all $(x, y) \in G \times G$; and

(K_5): (Approximate identity) $\lim_{t \to 0+} \int_G k(t, x, y) f(y) d\mu(y) = f(x)$ in the L^2-norm for all $f \in L^2(G)$.

(Note that integration spaces always refer to the measure μ.) In this chapter we shall always assume that the heat kernel k satisfies (K_1)-(K_5), but several of our results depend on further estimates on k, which reflect the fractal structure and the heat diffusion properties of G.

Typically a heat kernel has an inverse power law behaviour in t and decays exponentially with the separation of the spatial arguments. In the following estimate, the exponent d_w is the *walk dimension*, which reflects the rate of transport of heat through G, and d_s, is the *spectral dimension* which turns out to give the asymptotic distribution of the eigenvalues of the associated Laplacian, so that

$$d_s = 2 \lim_{\lambda \to \infty} \log \#\{\text{eigenvalues of } \Delta \text{ less than } \lambda\} / \log \lambda.$$

(Note that this is weaker than the definition (3.12).) In general

$$\frac{d_s}{2} = \frac{d_f}{d_w},$$

where d_f is the fractal dimension, that is the Hausdorff dimension, of G, see [**5, 6, 10**]. We will need estimates:

(K_6) (Bounds) there exist constants $0 < c_2 \le c_1$ and $a_1, a_2 > 0$ such that

$$a_1 t^{-\frac{d_s}{2}} \exp\left(-c_1\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \le k(t,x,y) \le a_2 t^{-\frac{d_s}{2}} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right),$$

for all $t > 0$ and $(x,y) \in G \times G$.

(That the condition (K_6) is equivalent to the doubling condition and Harnack inequality was recently proven in [**19, 20**].)

Note that for the classical case where $G \equiv \mathbb{R}^n$ and μ is n-dimensional Lebesgue measure, there is the usual Gaussian heat kernel

$$k(t,x,y) = (2\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{t}\right)$$

with $d_f = d_s = n$ and $d_w = 2$.

For later estimates we will require a Hölder condition on the heat kernel:

(K_7): (Hölder condition) there exist $\nu \ge 1$ and $0 < \sigma \le 1$ such that

$$|k(t,x_2,y) - k(t,x_1,y)| \le B_0\, t^{-\nu} |x_2 - x_1|^{\sigma}$$

for all $t > 0$ and $x_1, x_2 \in G$.

When we come to consider strong solutions, we need to control the derivatives of k with respect to t:

(K_8): $\frac{\partial k}{\partial t}(t,x,y)$ exists with $\left|\frac{\partial k}{\partial t}(t,x,y)\right| \le c\, t^{-(1+d_s/2)}$

for all $t > 0$ and $x, y \in G$ for some $c > 0$.

Heat kernels have been constructed on several classes of fractals. The best-known instance is the unbounded Sierpínski gasket in $\mathbb{R}^n$, see Barlow and Perkins [**6**], where heat kernels are termed transition densities and studied from the probabilistic point of view. In this case (K_1)-(K_5) hold, together with (K_6) for $d_s = \frac{2\log(n+1)}{\log(n+3)} < 2$, $d_f = \frac{\log(n+1)}{\log 2}$ and $d_w = \frac{\log(n+3)}{\log 2}$ (see [**18**]), and (K_7) with $\nu = 1$ and $\sigma = d_w - d_f$, see [**2**]. Estimates of the form (K_6) hold for affine nested fractals, see [**14**]. For more general unbounded p.c.f. self-similar sets, only a weaker version of (K_6) is known, see [**22**].

Heat kernels exist for the Sierpínski carpet in $\mathbb{R}^n$, see [**2, 4, 7**], and for generalized Sierpínski carpets (where different patterns of squares or cubes are selected in the construction) for which, in particular, (K_6) holds [**5**]. For the Sierpínski carpet in $\mathbb{R}^2$, we have $d_s \approx 1.80$. For the Sierpínski carpet in $\mathbb{R}^n$ the heat kernel $k(t,x,y)$ is smooth in $t>0$ for all $x,y \in G$ and satisfies (K_8), see [5]. The existence and properties of heat kernels on other fractals is a topic of active research.

We assume that the initial data $\phi : G \to \mathbb{R}$ is measurable; in what follows ϕ will generally be non-negative, sometimes satisfying further integrability conditions. By a *(weak) solution* to (7.1)-(7.2) we mean a measurable $u : (0,\infty)\times G \to \mathbb{R}$ satisfying the integral equation

$$u(t,x) = \int_G k(t,x,y)\phi(y)d\mu(y) + \int_0^t d\tau \int_G k(t-\tau,x,y)u(\tau,y)^p d\mu(y). \tag{7.3}$$

Let $\{P_t, t>0\}$ be the family of linear operators associated with k, that is

$$P_t f(x) = \int_G k(t,x,y)f(y)d\mu(y); \tag{7.4}$$

thus $P_t\phi$ may be thought of as the solution of the linear equation $\partial u/\partial t = \Delta u$. From (K_1)-(K_3), $P_t : L^q(G) \to L^q(G)$ for all $1 \le q \le \infty$. In particular, $\{P_t, t>0\}$ is a family of symmetric operators on $L^2(G)$ which, by (K_4), possesses the semigroup property

$$P_{s+t} = P_s P_t. \tag{7.5}$$

The contraction property

$$\|P_t f\|_q \le \|f\|_q \text{ for all } f \in L^q(G) \text{ and all } t>0 \tag{7.6}$$

follows for all $1 \le q \le \infty$, using the weighted Hölder inequality and (K_3). Property (K_5) states that the family of operators $\{P_t\}$ is strongly continuous, that is

$$\lim_{t\to 0} \|P_t f - f\|_2 = 0.$$

In particular, this implies that there exists an infinitesimal generator Δ

$$\Delta f = \lim_{t\to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}(\Delta), \tag{7.7}$$

where $\mathcal{D}(\Delta)$ is the space of all functions $f \in L^2(G)$ such that the limit in (7.7) exists in the L^2 norm and is finite, see [17]. This definition of the Laplacian enables us to consider strong solutions of (7.1)-(7.2), that is (7.1) holds pointwise for $t>0$ and $x \in G$.

In Section 2 we consider the non-existence problem and show that, provided that the heat kernel satisfies (K_6), there are no non-negative global solutions to (7.3) if $p \le 1+\frac{2}{d_s}$, however small the initial data $\phi \not\equiv 0$, that is, solutions 'blow up' in a finite time. In Section 3 we show that non-negative global solutions exist if $p > 1+\frac{2}{d_s}$ and the initial data is small enough. In the classical situation, bounded solutions (7.3) are smooth in both x and t, see [15]. We can not expect such smoothness on fractals, but in Section 4 we prove that solutions in $L^\infty(G)\cap L^1(G)$ are Hölder continuous in x if the initial data is Hölder continuous. Moreover, given (K_8) the solution is differentiable with respect to t for almost all $t > 0$ for all $x \in G$, and so satisfies (7.1) at such points.

7.2. Non-existence of global solutions

Our aim in this section is to show using (K_6) that there are no non-negative global solutions to (7.3) if $p \le 1 + \frac{2}{d_s}$ for non-negative initial data $\phi \not\equiv 0$, however small. The estimate (7.8) plays a major role in the non-existence proof. Note that (K_5) and (K_6) are not required at this stage.

Proposition 7.1. *Assume that the heat kernel k satisfies (K_1)-(K_4). Let $\phi \ge 0$ be essentially bounded, and assume that there is a non-negative essentially bounded function $u(t,x)$ satisfying (7.3) in $(0,T) \times G$. Then*

$$t^{\frac{1}{p-1}} \int_G k(t,x,y)\phi(y)\, d\mu(y) \le B_1 \tag{7.8}$$

for all $t \in (0,T)$ and all $x \in G$, where B_1 is independent of T and ϕ.

Proof. We sketch the proof, following [**15**] and [**53**]. Since $u(t,x)$ satisfies (7.3), we get that

$$u(t,x) \ge P_t\phi(x)$$

with P_t given by (7.4). Using (7.3) again, it follows that

$$\begin{aligned}
u(t,x) &\ge \int_0^t d\tau \int_G k(t-\tau,x,y)u(\tau,y)^p\, d\mu(y) \\
&\ge \int_0^t d\tau \int_G k(t-\tau,x,y)[P_\tau\phi(y)]^p\, d\mu(y) \\
&\ge \int_0^t d\tau \Big\{ \int_G k(t-\tau,x,y)P_\tau\phi(y)\, d\mu(y) \Big\}^p \\
&= \int_0^t d\tau [P_{t-\tau}(P_\tau\phi)(x)]^p \\
&= \int_0^t d\tau [P_t\phi(x)]^p \\
&= t[P_t\phi(x)]^p,
\end{aligned}$$

where we have used the weighted Hölder inequality, (K_4) and (7.5). Repeating this procedure of substitution in (7.3) we obtain by induction that

$$u(t,x) \ge \frac{t^{1+p+\cdots+p^{k-1}}[P_t\phi(x)]^{p^k}}{(1+p)^{p^{k-2}}(1+p+p^2)^{p^{k-3}}\cdots(1+p+\cdots+p^{k-1})}. \tag{7.9}$$

To see this, suppose that (7.9) holds for some $k \geq 1$. By (7.3), we see that

$$\begin{aligned}
u(t,x) &\geq \int_0^t d\tau \int_G k(t-\tau,x,y)u(\tau,y)^p \, d\mu(y) \\
&\geq \int_0^t d\tau \Big\{ \int_G k(t-\tau,x,y)u(\tau,y) \, d\mu(y) \Big\}^p \\
&\geq \int_0^t d\tau \Big\{ \int_G k(t-\tau,x,y) \\
&\quad \frac{\tau^{1+p+\cdots+p^{k-1}}[P_\tau\phi(y)]^{p^k}}{(1+p)^{p^{k-2}}(1+p+p^2)^{p^{k-3}}\cdots(1+p+\cdots+p^{k-1})} \, d\mu(y) \Big\}^p \\
&\geq \int_0^t \frac{\tau^{p+p^2+\cdots+p^k}}{(1+p)^{p^{k-1}}(1+p+p^2)^{p^{k-2}}\cdots(1+p+\cdots+p^{k-1})^p} \\
&\quad [P_{t-\tau}(P_\tau\phi(x))]^{p^{k+1}} \, d\tau \\
&= \frac{t^{1+p+\cdots+p^k}[P_t\phi(x)]^{p^{k+1}}}{(1+p)^{p^{k-1}}(1+p+p^2)^{p^{k-2}}\cdots(1+p+\cdots+p^k)},
\end{aligned}$$

which implies that (7.9) holds for $k+1$.

Therefore, for all $k \geq 1$,

$$t^{\frac{p^k-1}{(p-1)p^k}} P_t\phi(x) \leq u(t,x)^{\frac{1}{p^k}} \prod_{j=2}^{\infty} (1+p+\cdots+p^{j-1})^{\frac{1}{p^j}}. \tag{7.10}$$

Since

$$\log \prod_{j=2}^{\infty} (1+p+\cdots+p^{j-1})^{\frac{1}{p^j}} \leq \sum_{j=2}^{\infty} \frac{1}{p^j} \log\left(jp^j\right) < \infty,$$

the estimate (7.8) follows on letting $k \to \infty$ in (7.10). ■

We now prove the non-existence of global solutions if $p \leq 1 + \frac{2}{d_s}$, that is any solutions 'blow up' or become unbounded in a finite time. In the case of $p < 1 + \frac{2}{d_s}$, this is an easy consequence of (7.8) and only requires the left hand inequality of (K_6).

Theorem 7.2. *Suppose that k satisfies (K_1)-(K_4) and (K_6). If $p \leq 1 + \frac{2}{d_s}$ then (7.3) has no non-negative essentially bounded global solutions if $\phi(x) \geq 0$ and $\phi(x) \not\equiv 0$.*

Proof. By (K_6), we have

$$\begin{aligned}
&\liminf_{t\to\infty} t^{\frac{d_s}{2}} \int_G k(t,x,y)\phi(y) \, d\mu(y) \\
&\geq a_1 \liminf_{t\to\infty} \int_G \exp\left(-c_1 \left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \phi(y) \, d\mu(y) \\
&\geq B_2, \qquad (7.11)
\end{aligned}$$

where $B_2 = 1$ if $\|\phi\|_1 = \int_G \phi(y)\, d\mu(y) = +\infty$ and $B_2 = a_1\|\phi\|_1$ if $\|\phi\|_1 < \infty$. Combining with (7.8) and (7.11), this requires that, for some $B_3 > 0$, we have $t^{\frac{d_s}{2}-\frac{1}{p-1}} \geq B_3$ for all large t, which is impossible if $p < 1 + \frac{2}{d_s}$.

We now consider the more delicate case of $p = 1 + \frac{2}{d_s}$. Then (7.8) becomes

$$t^{\frac{d_s}{2}} \int_G k(t,x,y)\phi(y)\, d\mu(y) \leq B_1$$

which using the left hand side of $(K6)$ gives

$$\int_G \phi(y)\, d\mu(y) \leq B_4 < \infty, \tag{7.12}$$

where $B_4 = B_1/a_1$. Observe that for any $t_0 > 0$, $v(t,x) \equiv u(t+t_0,x)$ is a solution to (7.3) with initial data $\psi(x) = u(t_0,x)$. Repeating the above procedure we have that

$$\int_G u(t,y)\, d\mu(y) \leq B_5, \quad \text{for all } t > 0, \tag{7.13}$$

for some $B_5 > 0$.

Next we claim that for all $t_0 > 0$, there exists $\epsilon > 0$ and $b_1 > 0$, depending on t_0 and ϕ, such that

$$u(t_0,x) \geq b_1 k(\epsilon,x,0) \tag{7.14}$$

for all $x \in G$. To see this, let $\varrho_1 = \frac{1}{d_w-1}$ and $\varrho_2 = \frac{d_w}{d_w-1}$. By (K_6),

$$k(\epsilon,x,0)a_2^{-1}\epsilon^{\frac{d_s}{2}} \exp\left(c_2\frac{|x|^{\varrho_2}}{\epsilon^{\varrho_1}}\right) \leq 1$$

and thus, using (7.3) and (K_6),

$$\begin{aligned} u(t_0,x) &\geq \int_G k(t_0,x,y)\phi(y)\, d\mu(y) \\ &\geq a_1 t_0^{-\frac{d_s}{2}} \int_G \exp\left(-c_1\frac{|x-y|^{\varrho_2}}{t_0^{\varrho_1}}\right)\phi(y)\, d\mu(y) \\ &\geq \frac{a_1}{a_2}\left(\frac{\epsilon}{t_0}\right)^{\frac{d_s}{2}} k(\epsilon,x,0)\int_G \exp\left(\frac{c_2}{\epsilon^{\varrho_1}}|x|^{\varrho_2} - \frac{c_1}{t_0^{\varrho_1}}|x-y|^{\varrho_2}\right)\phi(y)\, d\mu(y). \qquad (7.15) \end{aligned}$$

Note that for $\varrho_2 \geq 1$

$$|x-y|^{\varrho_2} \leq 2^{\varrho_2-1}(|x|^{\varrho_2}+|y|^{\varrho_2})$$

for all $x,y \in \mathbb{R}$, so

$$\begin{aligned} \frac{c_2}{\epsilon^{\varrho_1}}|x|^{\varrho_2} &- \frac{c_1}{t_0^{\varrho_1}}|x-y|^{\varrho_2} \\ &\geq \left(\frac{c_2}{\epsilon^{\varrho_1}} - 2^{\varrho_2-1}\frac{c_1}{t_0^{\varrho_1}}\right)|x|^{\varrho_2} - 2^{\varrho_2-1}\frac{c_1}{t_0^{\varrho_1}}|y|^{\varrho_2} \\ &= -2^{\varrho_2-1}\frac{c_1}{t_0^{\varrho_1}}|y|^{\varrho_2} \end{aligned}$$

for all x, y, if ϵ is chosen to satisfy $c_2\epsilon^{-\varrho_1} = 2^{\varrho_2-1}c_1t_0^{-\varrho_1}$. Hence (7.14) follows on taking

$$b_1 = \frac{a_1}{a_2}\left(\frac{\epsilon}{t_0}\right)^{\frac{d_s}{2}} \int_G \exp\left(-2^{\varrho_2-1}\frac{c_1}{t_0^{\varrho_1}}|y|^{\varrho_2}\right)\phi(y)\,d\mu(y)$$

in (7.15).

To complete the proof when $p = 1 + \frac{2}{d_s}$, we consider $v(t,x) \equiv u(t+1,x)$. Clearly $v(t,x)$ satisfies (7.3) with the initial data $u(1,x)$ in place of $\phi(x)$. By (7.3), (7.14) and (K_4),

$$\begin{aligned} v(t,x) &\geq \int_G k(t,x,y)u(1,y)\,d\mu(y) \\ &\geq b_1\int_G k(t,x,y)k(\epsilon,y,0)\,d\mu(y) \\ &= b_1k(t+\epsilon,x,0) \end{aligned}$$

and so by (7.3)

$$\begin{aligned} \int_G v(t,x)\,d\mu(x) &\geq \int_G d\mu(x)\int_0^t d\tau\int_G k(t-\tau,x,y)v(\tau,y)^p\,d\mu(y) \\ &= \int_0^t d\tau\int_G v(\tau,y)^p\,d\mu(y) \\ &\geq b_1^p\int_0^t d\tau\int_G k(\tau+\epsilon,y,0)^p\,d\mu(y). \end{aligned} \tag{7.16}$$

Using (K_6) twice with $p = 1 + \frac{2}{d_s}$, it follows that

$$\begin{aligned} &k(\tau+\epsilon,y,0)^p \geq a_1^p(\tau+\epsilon)^{-pd_s/2}\exp\left(-pc_1|y|^{\varrho_2}/(\tau+\epsilon)^{\varrho_1}\right) \\ &= a_1^p\left(\frac{c_2}{pc_1}\right)^{d_s/2\varrho_1}(\tau+\epsilon)^{-1}\left[\frac{\tau+\epsilon}{(pc_1c_2^{-1})^{\frac{1}{\varrho_1}}}\right]^{-d_s/2}\exp\left(-\frac{c_2|y|^{\varrho_2}}{c_2(\tau+\epsilon)^{\varrho_1}/pc_1}\right) \\ &\geq \frac{a_1^p}{a_2}\left(\frac{c_2}{pc_1}\right)^{d_s/2\varrho_1}(\tau+\epsilon)^{-1}k\left((\tau+\epsilon)(pc_1c_2^{-1})^{-\frac{1}{\varrho_1}},y,0\right) \end{aligned}$$

which combines with (7.16) to yield

$$\int_G v(t,x)\,d\mu(x) \geq B_6\int_0^t(\tau+\epsilon)^{-1}\,d\tau \to \infty$$

as $t \to \infty$, where $B_6 > 0$. This contradicts (7.13), proving the theorem for $p = 1+\frac{2}{d_s}$.
■

The above theorem applies, for example, to the unbounded Sierpínski gasket and to generalised Sierpínski carpets.

7.3. Existence of non-negative global solutions

We now consider the situation where $p > 1 + \frac{2}{d_s}$ and show that if the initial data is sufficiently small, then (7.3) possesses a non-negative global solution.

We prove the following general existence result which we then apply in our situation, again adapting [**15, 52**].

Theorem 7.3. *Suppose k satisfies (K_1)-(K_5). Let $p > 1$ and $1 \le q < \infty$. Let $\phi \ge 0$ with $\phi \in L^q(G)$ and satisfying*

$$\int_0^\infty \|P_\tau \phi\|_\infty^{p-1} d\tau \le (p-1)^{-1}, \tag{7.17}$$

where P_τ is given by (7.4). Then (7.3) possesses a non-negative global solution u with $u \in L^\infty((0,T), L^q(G))$ for all $T > 0$, where $L^\infty((0,T), L^q(G))$ is the space consisting of all functions u such that

$$\|u(t)\|_q^q \equiv \int_G |u(t,x)|^q \, d\mu(x) \in L^\infty(0,T).$$

Proof. The proof follows [**52**]. We define $b : [0,\infty) \to \mathbb{R}$ by

$$b(t)^{-(p-1)} = 1 - (p-1) \int_0^t \|P_\tau \phi\|_\infty^{p-1} d\tau.$$

Using (7.17), we get that $b(0) = 1$ and $b'(t) = b(t)^p \|P_t \phi\|_\infty^{p-1}$. Thus b satisfies the integral equation

$$b(t) = 1 + \int_0^t b(\tau)^p \|P_\tau \phi\|_\infty^{p-1} d\tau. \tag{7.18}$$

Now let $u : [0,\infty) \times G \to [0,\infty]$ be any measurable function such that

$$P_t\phi(x) \le u(t,x) \le b(t) P_t \phi(x) \text{ for all } t \ge 0 \text{ and } x \in G,$$

and define

$$\mathcal{F}u(t,x) = P_t\phi(x) + \int_0^t (P_{t-\tau} u^p)(\tau, x) d\tau. \tag{7.19}$$

Then for $t \ge 0$ and $x \in G$, using (7.4), (7.5) and (7.18),

$$\begin{aligned} \mathcal{F}u(t,x) &\le P_t\phi(x) + \int_0^t d\tau \int_G k(t-\tau,x,y) b(\tau)^p [P_\tau \phi(y)]^p d\mu(y) \\ &\le P_t\phi(x) + \int_0^t b(\tau)^p \|P_\tau\phi\|_\infty^{p-1} d\tau \int_G k(t-\tau,x,y) P_\tau\phi(y) d\mu(y) \\ &= b(t) P_t \phi(x). \end{aligned}$$

Therefore,

$$P_t\phi(x) \le \mathcal{F}u(t,x) \le b(t) P_t\phi(x), \text{ for all } t \ge 0 \text{ and } x \in G.$$

Define $u_0(t,x) = P_t\phi(x)$ and $u_{m+1}(t,x) = \mathcal{F}u_m(t,x)$ for $m = 0,1,2,\ldots$. Using (7.19) and induction, for all $t > 0$ and $x \in G$, the sequence $\{u_m(t,x)\}$ is non-decreasing in m, and

$$P_t\phi(x) \le u_m(t,x) \le b(t)P_t\phi(x) \quad \text{for all } m \ge 0. \tag{7.20}$$

Thus there exists a measurable function $u(t,x)$ such that for all $t > 0$ and $x \in G$

$$\lim_{m\to\infty} u_m(t,x) = u(t,x) \in [0,\infty],$$

with

$$P_t\phi(x) \le u(t,x) \le b(t)P_t\phi(x), \quad t > 0, x \in G. \tag{7.21}$$

Using the monotone convergence theorem, we have

$$\lim_{m\to\infty} \int_0^t d\tau \int_G k(t-\tau,x,y)u_m(\tau,y)^p d\mu(y) = \int_0^t d\tau \int_G k(t-\tau,x,y)u(\tau,y)^p d\mu(y)$$

for all $t \ge 0$ and $x \in G$, and thus $u(t,x)$ satisfies (7.3) on taking the limit as $m \to \infty$ in

$$\begin{aligned} u_{m+1}(t,x) &= \mathcal{F}u_m(t,x) \\ &= P_t\phi(x) + \int_0^t d\tau \int_G k(t-\tau,x,y)u_m(\tau,y)^p d\mu(y). \end{aligned}$$

From (7.21)

$$\|u(t,.)\|_q \le b(t)\|P_t\phi\|_q \le b(t)\|\phi\|_q$$

by (7.6), so since $b(t)$ is bounded on $[0,T]$ we get $u \in L^\infty((0,T),L^q(G))$ for all $T > 0$. ■

We can immediately apply this result to bounded data.

Corollary 7.4. *Let k satisfy (K_1)-(K_6), and let $p > 1 + \frac{2}{d_s}$. Given $\gamma > 0$ there exists $\delta > 0$ such that, if*

$$0 \le \phi(x) \le \delta k(\gamma,x,0)$$

for all $x \in G$, then (7.3) has a non-negative bounded global solution. In particular, this will be the case if ϕ has compact support and $\sup_{x\in G}\phi(x)$ is sufficiently small.

Proof. Using (K_4) and (K_6),

$$\begin{aligned} \|P_t\phi\|_\infty &= \sup_{x\in G}\int_G k(t,x,y)\phi(y)d\mu(y) \\ &\le \delta \sup_{x\in G} k(t+\gamma,x,0) \\ &\le a_2\delta(t+\gamma)^{-d_s/2}, \end{aligned}$$

and so

$$\int_0^\infty \|P_t\phi\|_\infty^{p-1}dt \le (a_2\delta)^{p-1}\int_0^\infty (t+\gamma)^{-(p-1)d_s/2}dt \le (p-1)^{-1}$$

for δ small enough, since $p > 1 + 2/d_s$. Therefore, Theorem 7.3 implies the existence of a non-negative global solution. ■

We use the Marcinkiewicz interpolation theorem to apply Theorem 7.3 with an alternative condition on the initial data. Let D_1 and D_2 be linear spaces of measurable functions on two σ-finite measure spaces with measures μ_1 and μ_2 respectively. We say that a mapping $H : D_1 \to D_2$ is *subadditive* if

$$|H(f_1 + f_2)(x)| \le |H(f_1)(x)| + |H(f_2)(x)|$$

for all f_1 and f_2 in D_1 and μ_2-almost all x. For $1 \le r \le \infty$ and $1 \le s < \infty$, we say that H is of *weak type* (r, s) if there exists a constant A such that

$$\lambda(\varrho_3) \le \left(\frac{A\|f\|_r}{\varrho_3}\right)^s$$

for all $f \in L^r(\mu_1) \cap D_1$, where $\lambda(\varrho_3) = \mu_2\{x : |H(f)(x)| > \varrho_3\}$ is the distribution function of Hf, and $\|f\|_r = \left(\int |f(x)|^r d\mu_1(x)\right)^{\frac{1}{r}}$.

Proposition 7.5. *(Marcinkiewicz interpolation theorem) Suppose that H is a subadditive map of weak type (r_i, s_i) where $1 \le r_i \le s_i \le \infty$ for both $i = 1, 2$, and with $s_1 \ne s_2$. Then for all $\theta \in (0, 1)$, H is of strong type (r_θ, s_θ), where*

$$\frac{1}{r_\theta} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2} \quad \text{and} \quad \frac{1}{s_\theta} = \frac{1-\theta}{s_1} + \frac{\theta}{s_2},$$

that is there exists a constant A_0 such that

$$\|H(f)\|_{s_\theta} \le A_0 \|f\|_{r_\theta} \quad \text{for all } f \in D_1.$$

Proof. See [48, p184]. ■

Corollary 7.6. *Let k satisfy (K_1)-(K_6), and let $d_s \le 2$ and $p > 1 + \frac{2}{d_s}$. If $\|\phi\|_{d_s(p-1)/2}$ is sufficiently small then (7.3) has a non-negative global solution in $L^\infty((0,T), L^{d_s(p-1)/2}(G))$ for all $T > 0$, where $L^\infty((0,T), L^{d_s(p-1)/2}(G))$ consists of all functions u such that $\|u(t,.)\|_{d_s(p-1)/2} \in L^\infty(0,T)$.*

Proof. Fix $1 \le r \le \infty$. We consider maps $H : L^r(G) \to C(\mathbb{R})$ defined by

$$H\phi(t) = \|P_t\phi\|_\infty \quad \text{for } t > 0, \text{ where } \phi \in L^r(G).$$

Clearly H is subadditive. Moreover,

$$\begin{aligned} H\phi(t) &= \sup_{x\in G}\left|\int_G k(t,x,y)\phi(y)d\mu(y)\right| \\ &\le \sup_{x\in G}\left|\int_G k(t,x,y)\phi(y)^r d\mu(y)\right|^{\frac{1}{r}} \\ &\le a_2^{\frac{1}{r}} t^{-\frac{d_s}{2r}} \|\phi\|_r, \end{aligned}$$

by virtue of the weighted Hölder inequality and (K_6). Thus $H\phi(t) > \varrho_3$ implies that

$$t \le \left(\frac{a_2^{\frac{1}{r}}\|\phi\|_r}{\varrho_3}\right)^{\frac{2r}{d_s}}.$$

Thus H is of weak-type (r,s) whenever $1 \le r \le \infty$ and $s = 2r/d_s$. By Proposition 7.5, H is of strong-type (r,s) whenever $1 < r < \infty$ and $s = 2r/d_s$, that is there is a $B_7 > 0$ such that

$$\left(\int_0^\infty \|P_t\phi\|_\infty^s dt\right)^{\frac{1}{s}} \le B_7 \|\phi\|_r \quad \text{for all } \phi \in L^r.$$

Letting $r = d_s(p-1)/2 > 1$ we have $s = 2r/d_s > 1$ since $d_s \le 2$, so

$$\int_0^\infty \|P_t\phi\|_\infty^{p-1} dt \le B_7^{p-1} \|\phi\|_{d_s(p-1)/2}^{p-1} \le (p-1)^{-1}$$

if $\|\phi\|_{d_s(p-1)/2}$ is sufficiently small, giving (7.17). The conclusion follows from Theorem 7.3. ■

By our introductory remarks, Corollaries 7.4 and 7.6 imply that there exist non-negative solutions to (7.3) on the Sierpínski gasket in $\mathbb{R}^n$ and the Sierpínski carpet in $\mathbb{R}^2$ if $p > 1 + 2/d_s$ and the initial data is small in an appropriate sense.

7.4. Regularity properties of global solutions

In this section we discuss regularity properties of global solutions. Let $u : [0,\infty) \times G \to \mathbb{R}$ be a non-negative global solution to (7.3). We show that if the initial data is Hölder continuous, then $u(t,x)$ is Hölder continuous in x uniformly for $t \in (0,T)$ for all $T > 0$. We shall also show that $\frac{\partial u}{\partial t}(t,x)$ exists and satisfies (7.1) for almost every $t > 0$ for all $x \in G$, where the Laplacian Δ is viewed as the infinitesimal generator (7.7).

We first require an estimate for radial integrals which depends on local 'fractal' properties of the measure μ. We write $B_r(x)$ for the closed ball of centre x and radius r.

Proposition 7.7. *Let G be a closed (not necessarily bounded) subset of $\mathbb{R}^n$ and μ a Borel measure supported by G such that for some $d > 0$ and $B_8 > 0$*

$$\mu(B_r(x)) \le B_8\, r^d \quad (r > 0, x \in G). \tag{7.22}$$

Suppose that $f : [0,\infty) \to [0,\infty)$ is C_1 with $f(r) = o(r^{-d})$ as $r \to \infty$. Then

$$\int_G f(|y-x|)d\mu(y) \le B_8 \int_0^\infty r^d |f'(r)| dr, \quad (x \in G). \tag{7.23}$$

Proof. Let $m(r) = \mu(B_r(x))$. Then $m(r)$ is non-decreasing in r and continuous on the right. Therefore, for all $x \in G$, using that $m(0) = 0$ and $f(r) = o(r^{-d})$,

$$\begin{aligned}
\int_G f(|y-x|)d\mu(y) &\leq \int_0^\infty f(r)dm(r) \\
&= m(r)f(r)\Big|_0^\infty - \int_0^\infty f'(r)m(r)dr \\
&= -\int_0^\infty f'(r)m(r)dr \\
&\leq B_8 \int_0^\infty r^d |f'(r)|dr
\end{aligned}$$

using (7.22). ∎

Note that many measures satisfying (7.22) exist. For example, if μ is the restriction of d-dimensional Hausdorff measure to a (bounded or unbounded) self-similar set G of Hausdorff dimension d, then (7.22) is satisfied, see **[9, 10]**. In particular this applies to the Sierpínski gaskets and generalised carpets with $d = d_f$.

Corollary 7.8. *Let G be an unbounded subset of $\mathbb{R}^n$ supporting a measure μ which satisfies* (7.22). *Then given $\varrho_1, \varrho_2, \lambda, c_2 > 0$, there exists B_9 such that*

$$\int_G |y-x|^\lambda \exp\left(-c_2 \frac{|y-x|^{\varrho_2}}{t^{\varrho_1}}\right) d\mu(y) \leq B_9\, t^{(\lambda+d)\varrho_1/\varrho_2} \tag{7.24}$$

for all $t > 0$ and all $x \in G$.

Proof. For $t > 0$, take

$$f(r) = r^\lambda \exp\left(-c_2 \frac{r^{\varrho_2}}{t^{\varrho_1}}\right).$$

Clearly $f(r) = o(r^{-d})$ and

$$|f'(r)| \leq \left(\lambda + c_2\varrho_2 \frac{r^{\varrho_2}}{t^{\varrho_1}}\right) r^{\lambda-1} \exp\left(-c_2 \frac{r^{\varrho_2}}{t^{\varrho_1}}\right).$$

Using (7.23), it follows that

$$\begin{aligned}
&\int_G |y-x|^\lambda \exp\left(-c_2 \frac{|y-x|^{\varrho_2}}{t^{\varrho_1}}\right) d\mu(y) \\
&\leq B_8 \int_0^\infty r^{d+\lambda-1} \left(\lambda + c_2\varrho_2 \frac{r^{\varrho_2}}{t^{\varrho_1}}\right) \exp\left(-c_2 \frac{r^{\varrho_2}}{t^{\varrho_1}}\right) dr \\
&= B_8 t^{(\lambda+d)\varrho_1/\varrho_2} \int_0^\infty s^{d+\lambda-1}(\lambda + c_2\varrho_2 s^{\varrho_2}) \exp\left(-c_2 s^{\varrho_2}\right) ds,
\end{aligned}$$

giving (7.24) since this integral is finite and independent of t. ∎

Theorem 7.9. (Hölder continuity) *Assume that the measure μ on G satisfies* (7.22), *where $d = d_f$ is the Hausdorff dimension of G. Suppose that the heat kernel*

k satisfies (K_1)-(K_7) and suppose the initial data $\phi \in L^1(G)$ is Hölder continuous with exponent $\lambda \in (0,1]$, that is

$$|\phi(x_2) - \phi(x_1)| \le B_{10}|x_2 - x_1|^\lambda \quad \text{for all } x_1, x_2 \in G \tag{7.25}$$

for some B_{10}. Let $T > 0$ and let u be a non-negative global solution to (7.3) that is bounded on $(0,T) \times G$ with $\|u(t,.)\|_1$ bounded on $(0,T)$. Then $u(t,x)$ is Hölder continuous in x, with exponent $\gamma = \frac{\lambda\sigma}{\lambda + \nu d_w}$, uniformly for $t \in (0,T)$, where d_w is the walk dimension; that is

$$|u(t,x_2) - u(t,x_1)| \le B_{11}|x_2 - x_1|^\gamma \tag{7.26}$$

for all $t \in (0,T)$ and all $x_1, x_2 \in G$ where B_{11} depends on T.

Proof. We first prove that

$$u_0(t,x) \equiv P_t\phi(x) = \int_G k(t,x,y)\phi(y)d\mu(y) \tag{7.27}$$

is Hölder continuous in x uniformly for $t > 0$. To see this, by (K_7) we have that

$$\begin{aligned} |u_0(t,x_2) - u_0(t,x_1)| &= \left|\int_G (k(t,x_2,y) - k(t,x_1,y))\phi(y)d\mu(y)\right| \\ &\le B_0 t^{-\nu}|x_2 - x_1|^\sigma \|\phi\|_1 \\ &\le B_0\|\phi\|_1|x_2 - x_1|^{\sigma - \nu s} \qquad (7.28) \end{aligned}$$

if $t \ge |x_2 - x_1|^s$, for all $s > 0$.

On the other hand, from (7.25) and (K_6), taking $\varrho_1 = 1/(d_w - 1)$, $\varrho_2 = d_w/(d_w - 1)$ in (7.24) and recalling that $d_s/2 = d_f/d_w$,

$$\begin{aligned} &\int_G k(t,x,y)|\phi(y) - \phi(x)|d\mu(y) \\ &\le a_2 B_{10} t^{-\frac{d_s}{2}} \int_G |y - x|^\lambda \exp\left(-c_2\frac{|y-x|^{\varrho_2}}{t^{\varrho_1}}\right) d\mu(y) \\ &\le a_2 B_9 B_{10} t^{\frac{\lambda}{d_w}}. \end{aligned}$$

Therefore, if $t \le |x_2 - x_1|^s$, it follows from (7.25) that for all $t > 0$ and all $x_1, x_2 \in G$,

$$\begin{aligned} &|u_0(t,x_2) - u_0(t,x_1)| \\ &= \left|\int_G k(t,x_2,y)(\phi(y) - \phi(x_2))d\mu(y)\right. \\ &\quad \left. - \int_G k(t,x_1,y)(\phi(y) - \phi(x_1))d\mu(y) + (\phi(x_2) - \phi(x_1))\right| \\ &\le 2a_2 B_9 B_{10} t^{\frac{\lambda}{d_w}} + B_{10}|x_2 - x_1|^\lambda \\ &\le (B_{10} + 2a_2 B_9 B_{10})(|x_2 - x_1|^\lambda + |x_2 - x_1|^{\frac{s\lambda}{d_w}}). \end{aligned}$$

This combines with (7.28) to give

$$\begin{aligned} |u_0(t,x_2)-u_0(t,x_1)| &\le B_{12}(|x_2-x_1|^{\sigma-\nu s}+|x_2-x_1|^{\lambda}+|x_2-x_1|^{\frac{s\lambda}{d_w}}) \\ &\le 3B_{12}|x_2-x_1|^{\frac{\lambda\sigma}{\lambda+\nu d_w}} \end{aligned} \tag{7.29}$$

for all $t>0$ and all $x_1,x_2\in G$ with $|x_2-x_1|\le 1$, by taking s such that $\sigma-\nu s = s\lambda/d_w$, and noting that $\lambda > \lambda\sigma/(\lambda+\nu d_w)$, since $\sigma\le 1\le\nu$ and $d_w\ge 2$.

Now we consider

$$w(t,x)\equiv\int_0^t d\tau\int_G k(t-\tau,x,y)u(\tau,y)^p d\mu(y).$$

Fix $T>0$ and $0<\eta<t<T$. Since $u(t,x)$ is bounded in $(0,T)\times G$, it follows that

$$\int_{t-\eta}^t d\tau\int_G k(t-\tau,x,y)u(\tau,y)^p d\mu(y)\le B_{13}\eta \quad \text{for all } (t,x)\in(0,T)\times G$$

for some $B_{13}>0$. Assume first that $\nu>1$. Then for all $t\in(0,T)$ and $x_1,x_2\in G$, using (K_7),

$$\begin{aligned} |w(t,x_2)-w(t,x_1)| = \Big| &\int_{t-\eta}^t d\tau\int_G k(t-\tau,x_2,y)u(\tau,y)^p d\mu(y) \\ &-\int_{t-\eta}^t d\tau\int_G k(t-\tau,x_1,y)u(\tau,y)^p d\mu(y) \\ &+\int_0^{t-\eta} d\tau\int_G (k(t-\tau,x_2,y)-k(t-\tau,x_1,y))u(\tau,y)^p d\mu(y)\Big| \\ \le\ & 2B_{13}\eta + B_0\int_0^{t-\eta} d\tau\int_G |t-\tau|^{-\nu}|x_2-x_1|^{\sigma}u(\tau,y)^p d\mu(y) \\ \le\ & B_{14}(\eta+\eta^{1-\nu}|x_2-x_1|^{\sigma}), \end{aligned} \tag{7.30}$$

for some B_{14}. Since the left-hand side of (7.30) is independent of η, we may take $\eta=|x_2-x_1|^{\sigma/\nu}$ in (7.30) to get

$$\begin{aligned} |w(t,x_2)-w(t,x_1)| &\le B_{14}(|x_2-x_1|^{\sigma/\nu}+|x_2-x_1|^{\sigma+(1-\nu)\sigma/\nu}) \\ &\le 2B_{14}|x_2-x_1|^{\sigma/\nu} \end{aligned}$$

for all $t\in(0,T)$ and $x_1,x_2\in G$. Putting this estimate and (7.29) in the integral equation (7.3), and noting $\lambda\le 1<d_w$ gives the result when $|x_1-x_2|\le 1$, and this extends to all $x_1,x_2\in G$ since u is bounded. The case of $\nu=1$ is similar, with a logarithmic integral in (7.30). ∎

Next we show that, under certain conditions, for the bounded solution $u(t,x)$ of (7.3), $\frac{\partial u}{\partial t}(t,x)$ exists for almost every $t>0$ and all $x\in G$.

Theorem 7.10. (Lipschitz continuity) *Suppose that k satisfies (K_1)-(K_5) and (K_8). Assume that the initial data ϕ satisfies $\phi(x)\ge 0$, $\|\phi\|_1<\infty$, and*

$$|P_{t+\delta}\phi(x)-P_t\phi(x)|\le c_0\,\delta, \quad \text{for all } t>0 \text{ and } x\in G, \tag{7.31}$$

for some c_0, that is the solution $P_t\phi(x)$ of the corresponding linear equation for this initial data is Lipschitz continuous in t uniformly for x. Assume further that $u(t,x)$ is bounded, that is there exists a positive constant B_{15} such that

$$0 \le u(t,x) \le B_{15} \quad \text{for all } t > 0 \text{ and } x \in G, \tag{7.32}$$

Then for all $T > 0$, the solution $u(x,t)$ is uniformly Lipschitz on $(0,T)\times G$, that is $|u(t+\delta,x) - u(t,x)| \le A_1\ \delta$ for some A_1, so in particular $\frac{\partial u}{\partial t}(t,x)$ exists and is bounded for almost every $t > 0$, for all $x \in G$

Proof. Rewriting (7.3) as

$$u(t,x) = \int_G k(t,x,y)\phi(y)d\mu(y) + \int_0^t d\tau \int_G k(\tau,x,y)u(t-\tau,y)^p d\mu(y),$$

it follows from (7.31), (7.32) that

$$\begin{aligned} |u(t+\delta,x) - u(t,x)| &\le |P_{t+\delta}\phi(x) - P_t\phi(x)| \\ &\quad + \int_t^{t+\delta} d\tau \int_G k(\tau,x,y)u(\delta+t-\tau,y)^p d\mu(y) \\ &\quad + \int_0^t d\tau \int_G k(\tau,x,y)|u(\delta+t-\tau,y)^p - u(t-\tau,y)^p| d\mu(y) \\ &\le c_0\delta + B_{15}^p\ \delta \\ &\quad + p\ B_{15}^{p-1} \int_0^t d\tau \int_G k(\tau,x,y)|u(\delta+t-\tau,y) - u(t-\tau,y)| d\mu(y) \end{aligned}$$

for $t > 0, x \in G$, where we have used the inequality

$$|a^p - b^p| \le p\max(a^{p-1}, b^{p-1})|a-b|, \quad a \ge 0, b \ge 0.$$

Therefore, letting

$$f(t) = \sup_{x\in G}|u(t+\delta,x) - u(t,x)|, \quad t > 0,$$

we have that

$$f(t) \le (c_0 + B_{15}^p)\delta + pB_{15}^{p-1}\int_0^t f(t-\tau)d\tau.$$

Using Gronwall's inequality

$$f(t) \le (c_0 + B_{15}^p)\delta\exp(pB_{15}^{p-1}t), \quad t > 0,$$

so for $T > 0$,

$$|u(t+\delta,x) - u(t,x)| \le A_1\ \delta, \quad \text{for all } t \in (0,T) \text{ and } x \in G, \tag{7.33}$$

where $A_1 = (c_0 + B_{15}^p)\exp(pB_{15}^{p-1}T)$. Thus $u(t,x)$ is Lipschitz continuous in t for all $x \in G$, and so $\frac{\partial u}{\partial t}(t,x)$ exists for almost every $t > 0$ and for all $x \in G$. Clearly, by (7.33) $\frac{\partial u}{\partial t}(t,x)$ is bounded for $(t,x) \in (0,T)\times G$ for $T > 0$.■

Theorem 7.10 states that if, for given the initial data ϕ, the weak solution to the linear equation $\partial v/\partial t = \Delta v$ satisfies a Lipschitz condition, then so does the solution of the non-linear problem (7.3) for the same initial data. However, condition (7.31) may be difficult to verify directly. For one case, if ϕ is of the form

$$\phi(x) = \int_G k(\gamma, x, y)\psi(y)d\mu(y), \tag{7.34}$$

where $\psi(x) \geq 0$, $\|\psi\|_1 < \infty$ and $\gamma > 0$, then

$$P_t\phi(x) = \int_G k(t+\gamma, x, y)\psi(y)d\mu(y),$$

so by (K_8),

$$\begin{aligned} |P_{\delta+t}\phi(x) - P_t\phi(x)| &= \left| \int_G \psi(y)d\mu(y) \int_{t+\gamma}^{t+\delta+\gamma} \frac{\partial k}{\partial \tau}(\tau, x, y)d\tau \right| \\ &\leq \text{const } \gamma^{-(1+d_s/2)}\|\psi\|_1\delta, \end{aligned}$$

and thus ϕ satisfies (7.31).

Recall the definition (7.7) of the infinitesimal generator Δ of the semigroup $\{P_t, t > 0\}$ associated with the heat kernel k, which we take to be the Laplacian in (7.1). With this definition we can show that the weak solutions of (7.3) are strong solutions to (7.1)-(7.2), that is (7.1) holds pointwise at $(t, x) \in (0, \infty) \times G$, where the Laplacian is defined by (7.7).

Theorem 7.11. (Regularity) *Suppose that k satisfies satisfies (K_1)-(K_5) and (K_8). Let $u(t,x)$ be a non-negative bounded continuous solution of (7.3) for $t \in (0,T)$ and suppose that $\frac{\partial u}{\partial t}(t,x)$ exists for all $x \in G$ and $t \in (0,T)$. Moreover, suppose that $u(t,x)$, $\frac{\partial u}{\partial t}(t,x)$ are both uniformly $L^2(G)$-integrable for $t \in (0,T)$. Then*

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u(t,x)^p$$

for $t \in (0,T)$ and $x \in G$.

Proof. Since $u(t,x)$ satisfies (7.3) we have that, for $t_0 > 0$,

$$\begin{aligned} P_t u(t_0, x) &= \int_G k(t, x, y)u(t_0, y)d\mu(y) \\ &= \int_G k(t, x, y)\Big\{ \int_G k(t_0, y, z)\phi(z)d\mu(z) \\ &\qquad + \int_0^{t_0} d\tau \int_G k(t_0 - \tau, y, z)u(\tau, z)^p d\mu(z)\Big\} d\mu(y) \\ &= \int_G k(t + t_0, x, z)\phi(z)d\mu(z) \\ &\qquad + \int_0^{t_0} d\tau \int_G k(t + t_0 - \tau, x, z)u(\tau, z)^p d\mu(z) \end{aligned}$$

Using (7.3) again, it follows that

$$P_t u(t_0,x) - u(t_0,x) = \int_G (k(t+t_0,x,z) - k(t_0,x,z))\phi(z)d\mu(z)$$

$$+ \int_0^{t_0} d\tau \int_G (k(t+t_0-\tau,x,z) - k(t_0-\tau,x,z))u(\tau,z)^p d\mu(z). \tag{7.35}$$

On the other hand, by (7.3),

$$\begin{aligned} &u(t+t_0,x) - u(t_0,x) \\ &= \int_G (k(t+t_0,x,z) - k(t_0,x,z))\phi(z)d\mu(z) \\ &+ \int_0^{t_0} d\tau \int_G (k(t+t_0-\tau,x,z) - k(t_0-\tau,x,z))u(\tau,z)^p d\mu(z) \\ &+ \int_{t_0}^{t+t_0} d\tau \int_G k(t+t_0-\tau,x,z)u(\tau,z)^p d\mu(z). \end{aligned} \tag{7.36}$$

Combining (7.35) and (7.36),

$$\begin{aligned} P_t u(t_0,x) - u(t_0,x) &= u(t+t_0,x) - u(t_0,x) \\ &\quad - \int_{t_0}^{t+t_0} d\tau \int_G k(t+t_0-\tau,x,z)u(\tau,z)^p d\mu(z) \end{aligned} \tag{7.37}$$

for all $x \in G$ and $t > 0$. Since $u(t,x)$ is continuous and bounded, it follows by (K_5) that

$$\lim_{t\to 0} \frac{1}{t} \int_{t_0}^{t+t_0} d\tau \int_G k(t+t_0-\tau,x,z)u(\tau,z)^p d\mu(z) = u(t_0,x)^p$$

for each $x \in G$. Therefore, from (7.37) we deduce that

$$\lim_{t\to 0} \frac{1}{t}(P_t u(t_0,x) - u(t_0,x)) = \frac{\partial u}{\partial t}(t_0,x) - u(t_0,x)^p$$

for each $x \in G$. The limit here is pointwise, but using the uniform integrability of $u(t,x)$ and $\partial u/\partial t(t,x)$ and the dominated convergence theorem, the limit also exists in the L^2−norm, so the result follows by the definition (7.7) of the infinitesimal generator Δ. ■

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